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# Estimating dynamic discrete choice models with aggregate data: Properties of the inclusive value approximation

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## Abstract

We investigate the use of the inclusive value based approach for estimating dynamic discrete choice models of demand with aggregate data. The inclusive value sufficiency (IVS) approach approximates a multi-dimensional state space with a single "sufficient statistic" in order to mitigate the curse of dimensionality and tractability estimate model primitives. Although in widespread use, the conditions under which IVS is appropriate have not been examined. Theoretically, we show that the estimator is biased and inconsistent. We then use Monte Carlo simulations (of a simple model of dynamic durable goods adoption) to demonstrate the degree of bias associated with the inclusive value approximation estimator under an array of parameterizations and data generating processes. In our examination, we show that the estimator performs better when the discount factor is smaller and/or when the price sensitivity of the consumer is larger. Examining how the bias impacts economic quantities of interest, we find that the IVS method under estimates the true long-run own-price elasticities and over estimates the change in profits as prices change. Theses findings highlight the importance of correctly specifying how consumers form expectations. As a result, researchers should consider how to empirically support their assumption for the underlying consumer belief structure.


**Keywords** Dynamic structural models · Inclusive value

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## 1 Introduction

Dynamic Discrete choice (DDC) models are commonly used in marketing and economics, where agents (e.g. consumers) choose from among a limited set of mutually exclusive alternatives (e.g. products or brands). Moreover, many of these papers employ aggregate data because it is typically more commonly available from market research firms for a wide range of industries. Such data is specified as market shares for products or brands over a number of time periods, and for a number of different markets.

With models of dynamic durable good adoption, estimation is typically computationally demanding, as each product characteristic may span a continuous range, and the number of such observables may grow very large– increasing in the number of products, the number of product characteristics, and the number of markets. Most, if not all, models that use aggregate data have relied upon methods to reduce the state space to overcome the curse of dimensionality through the selection of ad hoc important state variables and markets with a small number of products, (Song and Chintagunta 2003) or through the use of a "sufficient statistic" to capture the relevant state.

The latter method developed in further detail in Melnikov (2013) and Gowrisankaran and Rysman (2012) is widely used in the literature with aggregate data (Carranza 2010; Schiraldi 2011; Derdenger and Kumar 2013; Weiergraeber 2017; Ho 2015, to cite a few) as well as individual data (Hendel and Nevo 2006). The method tractably allows for the estimation of a high dimensional problem, but it also allows the researcher a straight forward method to correct for price endogeneity–a widely accepted problem for models with aggregate data. Specifically, in order to mitigate the computational demands of consumers forming expectations of each product's characteristic evolution over time, Gowrisankaran and Rysman (2012) (G&R) assume consumers track the evolution of the inclusive value as the only state variable, where the *inclusive value* (McFadden 1974) represents the expected maximum utility of purchasing an "inside" good, i.e. excluding the no-purchase or outside option. This inclusive value captures in one variable all the population level observable and unobservable elements that could potentially be present in the state space. The underlying assumption of this approach is that consumers make choice decisions based upon how this market level inclusive value evolves, rather than on individual product level attributes. Thus, the inclusive value is treated as "sufficient" to capture the impact of all other variables. Despite its prevalence in the marketing and economics literature and its computational simplicity, the accuracy of the inclusive value assumption and how well it approximates the true underlying dynamics in which consumers track all state variables individually has not been carefully examined.

First, we theoretically show that the estimator is biased and inconsistent. We then use Monte Carlo simulations (of a simple model of dynamic durable goods adoption) to demonstrate the degree of bias associated with the inclusive value approximation estimator under an array of parameterizations and data generating processes. In our examination, the IVS estimator performs better when the discount factor is small ($\beta = .8$ vs $\beta = .95$) and/or when the price sensitivity of the consumer is large ($\alpha_p = -.3$ vs $\alpha_p = -.2$). We further analyze the bias the IVS generates with respect to short and long run elasticity measures. Within our Monte Carlo environment, results illustrate that the IVS method *underestimates the true long-run own-price* *elasticities* and *overestimates the change in profits* as prices change. These findings highlight the importance of correctly specifying how consumers form expectations about the underlying dynamics. We conclude that researchers should begin to empirically support their decisions about the underlying consumer belief structure in order to affirm the reader about the validity of their results.

## 2 Structural model

Dynamic discrete choice models of demand assume consumers are forward looking, and weigh a trade-off between making a purchase today versus the option value of waiting. Before entering the market, consumers consider numerous product and market characteristics that may affect their current and future purchase utilities, such as price, age of product, quality, etc.

The sequence of events is as follows: consumer $i \in \mathbf{I}$ considers whether or not to purchase any product from the available set with 0 representing the outside option. Thus, the choice set is $\mathbf{J_t} \subseteq \{0, 1, ...J\}$. In each period $t \in \mathbf{T}$, a consumer purchases or chooses not to purchase any product. Purchasing a product is a terminal action in our model, and once a purchase is made, the consumer has no active role in the market. The consumer decision process is thus equivalent to an optimal stopping problem with many available choices.

### 2.1 Consumer utility

Consumer $i$ determines in period $t$ whether or not to purchase any product $j$, by observing a vector of individual-level state variables $\vartheta_{i,t}$ specific to the consumer and time period. The state can be described as $\vartheta_{i,t} = (x_t, \mathbf{\xi_t}, \epsilon_{i,t})$, where $x_t$ is a matrix of *observed* market level states, $\xi_t$ is a vector of the *unobserved* product characteristics for each product (also called the unobserved population level states), and $\epsilon_{i,t}$ is the vector of individual choice-specific *idiosyncratic* private shocks, which are not observable to the researcher.

Typically, in a product choice model, we include all the product variables in the state space, $x_t = (x_{1t}, \ldots, x_{Jt})$ where $x_{jt} = \left(x_{jt}^c, p_{jt}\right)$, with $x_{jt}^c$ denoting a vector of observable product characteristics and $p_{jt}$ the price for choice $j$ in period $t$. The unobservable states or "structural errors" in the model are denoted:

$$\xi_t = (\xi_{1t}, \xi_{2t}, \ldots, \xi_{Jt})$$

where $\xi_{j,t}$ is a time-varying choice-specific variable that is unobservable (to the econometrician), typically thought of as a measure of functional or design quality. If the consumer does not purchase in period $t$, he receives a period utility of 0.

Denote the market-level states as $\Omega_t = (x_t, \xi_t)$, which includes both observable and unobservable states. Thus, the vector of state variables $\vartheta_{i,t} = (x_t, \mathbf{\xi_t}, \epsilon_{i,t}) = (\Omega_t, \epsilon_{it})$. When a consumer chooses to purchase product $j$ at time $t$ he receives a net flow utility in each of the following periods $\tau \geq t$

$$f_{j,\tau}\left(x_t^c, \xi_t\right) = \alpha_j + \alpha_x x_{j,t}^c + \xi_{j,t}.$$

Note that this flow utility in period $\tau$ is fixed at the time of purchase $t$ and depends on the observable and unobservable characteristics at $t$. Thus, when a consumer $i$ purchases $j$ at time $t$, his utility *during the purchase period t* is:

$$u_{it}(\Omega_t, \epsilon_{ijt}) = f_{j,t}\left(x_t^c, \xi_t\right) + \alpha_i^p p_{jt} + \epsilon_{ijt} \tag{1}$$

where $\alpha_i^p$ is the consumer price coefficient.

### 2.2 Dynamic decision problem

The consumer makes a trade-off between buying in the current period $t$ and waiting until next period to make a purchase. The crucial inter-temporal trade-off is in the consumer's expectation of how the state variables $x_t$ evolve in the future. For example, if the product characteristics (or price) are expected to improve over time, then the consumer has incentive to wait.

Consumer $i$ in period $t$ chooses from the set of choices $J_t$, which includes the option 0 to wait without purchasing any product. However, if the consumer purchases, he exits the market immediately upon purchase. A consumer's purchase period utility is impacted by the observable state vector $x_t$, the unobservable $\xi_t$ (both included in $\Omega_t$) as well as the idiosyncratic shocks as specified in Eq. 1.

For a consumer in the product market faced with a state $\Omega_t$ in period $t$, we can write the Bellman equation in terms of the value function $V(\Omega_t, \epsilon_t)$ as follows:

$$V_i(\Omega_t, \epsilon_t) = \max\left[\underbrace{\epsilon_{i0t} + \beta\, \mathbf{E}_{\Omega_{t+1}, \epsilon_{t+1}|\Omega_t}\left[V_i\left(\Omega_{t+1}, \epsilon_{t+1}\right)|\,\Omega_t\right]}_{\text{No Purchase}}, \quad \max_{j \in J_t \setminus \{0\}}\left[\underbrace{v_{i,j}\left(\Omega_t\right) + \epsilon_{ijt}}_{\text{Purchase j}}\right]\right]$$

where the first term within brackets is the present discounted utility associated with the decision to not purchase any product in period $t$. The choice of not purchasing in period $t$ provides zero flow utility per period, the realized value of an error term for option $j = 0$ in period $t$ and a term that captures expected future utility associated with choice $j = 0$ conditional on the current state being $\Omega_t$. This last term is the option value of waiting to purchase. The second term within brackets indicates the value associated with the purchase of a product. Given the fact that consumers exit the market after the purchase of any product a consumer's choice specific value function can be written as the sum of the current period $t$ utility and the stream of utilities in periods following purchase:[1]

$$v_{i,j}\left(\Omega_t\right) = \frac{1}{1-\beta} f_{j,t}\left(x_t^c, \xi_t\right) + \alpha_i^p p_{jt} \tag{2}$$

$$= \frac{1}{1-\beta}\left[\alpha_j + \alpha_x x_{j,t} + \xi_{j,t}\right] + \alpha_i^p p_{jt}. \tag{3}$$

[1] The below assumes a constant flow of utility after the purchase of a product, but this can be generalized to the case were flow utilities are time-varying (e.g. in the presence of complementary products).

We write the ex-ante value function $\overline{V}$, which represents the value of being in state $\Omega_t$ before the value of the shock $\epsilon_t$ is realized, as the expectation over the shocks:

$$\overline{V_i}(\Omega_t) = \int V_i(\Omega_t, \epsilon_t)\phi(\epsilon_t)d\epsilon_t.$$

where $\phi$ is the multivariate distribution of idiosyncratic errors across the choice set.

With assuming that the idiosyncratic errors $\epsilon$ are distributed as Type I extreme value random variables centered at zero, we can rewrite the Bellman equation in terms of the ex-ante value function as:

$$\overline{V_i}(\Omega_t) = \log\left(\sum_{j \in J_t} \exp\left(v_{i,j}(\Omega_t)\right)\right) = \log\left(\exp\left[\beta\, \mathbf{E}_{\Omega_{t+1}|\Omega_t}\left[\overline{V_i}(\Omega_{t+1})|\Omega_t\right]\right] + \sum_{j \in J_t \backslash \{0\}} \exp\left[v_{i,j}(\Omega_t)\right]\right),$$

which is obtained from the choice-specific value function of waiting, i.e. with $v_0(\Omega_t) = \beta\, \mathbf{E}_{\Omega_{t+1}|\Omega_t}\left[\overline{V_i}(\Omega_{t+1})\middle|\, \Omega_t\right]$. The market shares $s_{i,j}(\Omega_t)$ of choosing each $j \in \mathbf{J}$ given the state $\Omega_t$ can then be written in closed form as:

$$s_{i,j}(\Omega_t) = \frac{\exp\left(v_{i,j}(\Omega_t)\right)}{\sum_{j' \in J_t} \exp\left(v_{i,j'}(\Omega_t)\right)}.$$

### 2.3 The inclusive value sufficiency assumption (IVS)

The idea of an inclusive value works as follows. First, define and compute the *expected value of the maximum of utilities from the purchase choice set* (excluding the no-purchase option) as the inclusive value. Second, it is *assumed* that the inclusive value is sufficient to capture *all* the dynamic factors into one state variable. This assumption is termed *inclusive value sufficiency* (Gowrisankaran and Rysman 2012), and implies that all states with the same inclusive value term have the same value for the expected value function. Third, the inclusive value is modeled as evolving over time according to a specified process, typically AR(1), and assumes consumers have rational expectations regarding its evolution.

This inclusive value simplification ensures that the state space is tractable by dramatically reducing the state space to one dimension, defined as $\delta_i(\Omega_t)$:

$$\delta_i(\Omega_t) = E_\epsilon\left[\max_{j \in J \backslash 0} v_{i,j}(\Omega_t) + \epsilon_{jt}\right] = \log\left(\sum_{k \in \mathbf{J_t}} \exp\left(v_{i,k,t}\right)\right).$$

The Bellman equation can consequently be expressed in terms of the inclusive value, $\delta_{i,t}$:

$$\overline{V_i}\left(\delta_{i,t}\right) = \log\left(\underbrace{\exp\left(\delta_{i,t}\right)}_{\text{Purchase}} + \underbrace{\exp\left(\beta\, \mathbf{E}\left[\overline{V_i}(\delta_{i,t+1})|\delta_{i,t}\right]\right)}_{\text{No Purchase}}\right).$$

The evolution of the inclusive value is specified as evolving according to an $AR(1)$ process:

$$\delta_{i,t+1} = \gamma_{0,i} + \gamma_{1,i}\delta_{i,t} + \zeta_{i,t}$$

where $\zeta_{i,t}$ is normally distributed and is $iid$ across consumers and time periods. The individual-specific parameters $\gamma_{0,i}$ and $\gamma_{1,i}$ characterize the evolution of the inclusive value state, and yield a probability distribution for the future state, conditional on the current state.

### 2.4 Implications of the inclusive value sufficiency assumption

Although the benefit of this method is that it reduces the computational burden of estimating the model primitives, it comes with the cost that consumers are assumed to react identically to different types of changes only through the inclusive value. Thus, a new product introduction or increased product availability could have the same positive impact on the inclusive value as a price reduction. In the smartphone market, for example, the model might view the following as equivalent since they each improve the expected utility of the "best" choice option for consumers:

1. introduction of a new model of high-quality iPhone
2. introduction of multiple low-quality phones
3. price reduction for existing products on the market.
4. improvement in product characteristics of existing models (e.g. more memory capacity)

The problem with the approach is that a consumer's decision might be quite different under each of the above scenarios. We examine how accurately this assumption approximates the true data generating process for estimation: when does this approximation method work best and when might it not?

## 3 Evaluating the inclusive value assumption

We first present a theoretical analysis of the IVS estimator and follow with Monte Carlo simulations. The Monte Carlo data generating process (DGP) is created to reflect the essential feature of dynamic forward-looking models with consumers facing intertemporal tradeoffs between purchasing a product in the current period, compared to waiting for better product characteristics or prices. We use a simple model where there are no observable product characteristics and consumers value price and the unobservable characteristic. Consumers exit the market following a purchase, and continue if they have not made a purchase. It is important to note that the DGP ("true model") differs significantly from the IVS method. Specifically, the underlying model assumes consumers track a multidimensional state variable to form expectations, whereas the IVS method assumes that consumers track only the inclusive value. Thus, the IVS method might not be unable to fully capture a consumer's decision process due to this assumption.

## 4 Theoretical Properties of the Inclusive Value Approximation

In this subsection, we theoretically analyze the small and large sample properties of the IVS estimator.

**Proposition 1** *The IVS estimator is biased and inconsistent even when there is no price endogeneity.*

*Proof* Assume the data generating process follows from above with homogeneous consumer preferences. For sake of the proof, assume that the DGP value of $\mathbf{E}_{\Omega_{t+1}|\Omega_t}\left[\overline{V}\left(\Omega_{t+1}\right)|\Omega_t\right]$ is known.[2] Under this scenario the complex dynamic discrete choice model can be transformed into a linear model, where $\tilde{\xi}_{j,t} = \frac{\xi_{jt}}{1-\beta}$ and $\tilde{\alpha} = \frac{\alpha}{1-\beta}$ and $Y_\Omega$ is known.

$$Y_\Omega \equiv \log\left(\frac{s_{jt}}{s_{0t}}\right) + \beta\mathbf{E}_{\Omega_{t+1}|\Omega_t}\left[\overline{V}\left(\Omega_{t+1}\right)|\Omega_t\right] = \tilde{\alpha} + \alpha^p p_{jt} + \tilde{\xi}_{j,t}.$$

Define the first-stage reduced form relationship between the instrument $Z$ and endogenous variable $p$ as specified below with parameters $\varsigma$ and $\gamma$ and reduced form error $\nu$:

$$p_{jt} = \varsigma + \gamma Z_{jt} + \nu_{jt}$$

Denote the defined instrumental variable (IV) estimate of $\alpha^p$ as

$$\hat{\alpha}^p = \frac{Cov(\mathbf{Z}, \boldsymbol{Y_\Omega})}{Cov(\mathbf{Z}, \boldsymbol{p})}$$

Similarly, the IVS estimator can be specified via the DGP of $\mathbf{E}_{\Omega_{t+1}|\Omega_t}\left[\overline{V}\left(\Omega_{t+1}\right)|\Omega_t\right]$ *and* an approximation error, $\eta(\delta_t)$ where $\eta(\delta_\mathbf{t}) = \mathbf{E}_{\Omega_{t+1}|\Omega_t}\left[\overline{V}\left(\Omega_{t+1}\right)|\Omega_t\right] - \hat{\mathbf{E}}_{\delta_{t+1}|\delta_t}\left[\overline{V}\left(\delta_{t+1}\right)\middle|\,\delta_t\right]$ and $\hat{\mathbf{E}}_{\delta_{t+1}|\delta_t}\left[\overline{V}\left(\delta_{t+1}\right)\middle|\,\delta_t\right]$ is the IVS approximation-estimate of the expected value function. Under this setting, the model can be written as:

$$\log\left(\frac{s_{jt}}{s_{0t}}\right) = \tilde{\alpha}_{IVS} + \alpha^p_{IVS} p_{jt} + \tilde{\chi}_{j,t} - \beta \quad \hat{\mathbf{E}}_{\delta_{t+1}|\delta_t}\left[\overline{V}\left(\delta_{t+1}\right)\middle|\,\delta_t\right]$$

$$Y_{IVS} \equiv \log\left(\frac{s_{jt}}{s_{0t}}\right) + \beta \quad \hat{\mathbf{E}}_{\delta_{t+1}|\delta_t}\left[\overline{V}\left(\delta_{t+1}\right)\middle|\,\delta_t\right] = \tilde{\alpha}_{IVS} + \alpha^p_{IVS} p_{jt} + \tilde{\chi}_{j,t}$$

[2]This assumption is only for the sake of proving biasedness. However, note that if we were interested in proving unbiasedness, this argument would be problematic. We thank an anonymous reviewer for this point.

Importantly, the structural error in this model is $\chi_{j,t}$, which is different from the structural error in the true model as it encompasses the approximation error of the IVS method from the true model. With $\hat{\mathbf{E}}_{\delta_{t+1}|\delta_t}\left[\overline{V}(\delta_{t+1})\middle|\,\delta_t\right] = \mathbf{E}_{\Omega_{t+1}|\Omega_t}\left[\overline{V}(\Omega_{t+1})\,|\Omega_t\right] - \eta(\delta_t)$, the structural errors under the IVS estimator are related as :

$$\tilde{\chi}_{j,t} = \tilde{\xi}_{j,t} + \beta\eta(\delta_t)$$

The IVS estimate $\hat{\alpha}^p_{IVS}$ then equals:

$$\begin{aligned}
\hat{\alpha}^p_{IVS} &= \frac{Cov\,(Z, Y_{IVS})}{Cov(Z,p)} = \frac{Cov\,(Z, \tilde{\alpha}_0 + \alpha^p p + \tilde{\chi})}{Cov(Z,p)} \\
\hat{\alpha}^p_{IVS} &= \frac{Cov\left(Z, \tilde{\alpha}_0 + \alpha^p p + \tilde{\xi}_{j,t} + \beta\eta(\delta_t)\right)}{Cov(Z,p)} \\
&= \frac{Cov(Z, Y_\Omega) + Cov(Z, \beta\boldsymbol{\eta}(\boldsymbol{\delta_t}))}{Cov(Z,p)} \\
\hat{\alpha}^p_{IVS} &= \hat{\alpha}^p + \beta\frac{Cov\,(Z, \boldsymbol{\eta}(\boldsymbol{\delta_t}))}{Cov(Z,p)} \\
\mathrm{E}\left[\hat{\alpha}^p_{IVS}\right] &= \alpha^p + \underbrace{\beta\,\mathrm{E}\left[\frac{Cov\,(Z, \boldsymbol{\eta}(\boldsymbol{\delta_t}))}{Cov(Z,p)}\right]}_{\text{Bias}} \text{since } \mathrm{E}[\hat{\alpha}^p] = \alpha^p
\end{aligned}$$

Thus, we find that $\hat{\alpha}^p_{IVS}$ is biased above and beyond the small sample bias that may occur with an IV estimator.[3] The additional bias associated with this estimator is a function of $Cov\,(Z, \boldsymbol{\eta}(\boldsymbol{\delta_t}))$ where Z is an instrument for price given it is typically assumed that the $Cov\,(p, \xi) \neq 0$. As the covariance between the instrument and the approximation error increases in absolute value, the magnitude of that bias increases, holding all other things constant.[4]

The IVS estimator is also inconsistent since $plim_{N\to\infty}N^{-1}\left(Z^T\boldsymbol{\eta}(\boldsymbol{\delta_t})\right) \neq 0$.[5] By Slutsky's limit theorems, we can write

$$plim_{N\to\infty}\left[\hat{\alpha}^p_{IVS}\right] = \alpha^p + \frac{plim_{N\to\infty}N^{-1}\left(Z^T\xi\right)}{plim_{N\to\infty}N^{-1}\left(Z^Tp\right)} + \beta\frac{plim_{N\to\infty}N^{-1}\left(Z^T\boldsymbol{\eta}(\boldsymbol{\delta_t})\right)}{plim_{N\to\infty}N^{-1}\left(Z^Tp\right)}$$

$$plim_{N\to\infty}\left[\hat{\alpha}^p_{IVS}\right] = \alpha^p + \beta\frac{plim_{N\to\infty}N^{-1}\left(Z^T\boldsymbol{\eta}(\boldsymbol{\delta_t})\right)}{plim_{N\to\infty}N^{-1}\left(Z^Tp\right)}.$$

[3] Andrews and Armstrong (2017) illustrate that an IV estimator under certain conditions is an unbiased estimate of $\alpha^p$. Thus,$\mathrm{E}[\hat{\alpha}^p] = \alpha^p$.

[4] In our Monte Carlos simulations below we find that $Cov\,(Z, \boldsymbol{\eta}(\boldsymbol{\delta_t})) < 0$ leading to a negative bias associated with $\hat{\alpha}^p_{IVS}$ given $Cov(Z,p) > 0$. Additionally, note in small samples $\mathrm{E}\left[\frac{Cov(Z,\boldsymbol{\eta}(\boldsymbol{\delta_t}))}{Cov(Z,p)}\right] \neq \frac{\mathrm{E}[Cov(Z,\boldsymbol{\eta}(\boldsymbol{\delta_t}))]}{\mathrm{E}[Cov(Z,p)]}$. Thus, if the $\mathrm{E}[Cov\,(Z, \boldsymbol{\eta}(\boldsymbol{\delta_t}))] = 0$, the bias would still remain.

[5] In this case, we use the number of observations $N = |J| \times T$.

While $plim_{N\to\infty}N^{-1}\left(Z^T\xi\right)=0$, the challenge for the IVS estimator is finding an instrument for price $(Z)$ that is uncorrelated with the structural errors $(\xi)$ **and** the approximation error $\boldsymbol{\eta}(\boldsymbol{\delta_t})$ such that $plim_{N\to\infty}N^{-1}\left(Z^T\boldsymbol{\eta}(\boldsymbol{\delta_t})\right)=0$.

Consider the case without endogeneity, i.e. price $p_{j,t}$ is not correlated with $\xi_{j,t}$. We can use $p_{jt}$ as an instrument directly. In such a case, the term $Cov\left(P_{j,t},\boldsymbol{\eta}(\boldsymbol{\delta_t})\right)$ is not zero since the inclusive value $\delta_t$ depends on price $p_{j,t}$, implying the estimator is biased and inconsistent. This finding highlights that the IVS estimator generates bias estimates for exogeneous (product characteristics) variables as well. □

**Proposition 2** *In the case where there is an endogenous variable price present, there can be no valid instrument that can provide an unbiased and consistent estimate.*

*Proof* We consider here the endogenous price case. For $Z$ to be a potential instrument, it needs to be correlated with price and uncorrelated with the structural error $\chi_{j,t}$. Since $\chi_{j,t}=\xi_{j,t}+\beta\eta(\delta_t(p_t))$, any instrument $Z$ that is correlated with price will also be correlated with the error given $\eta(\delta_t(p_t))$. However, this leads to a problematic situation. We know from the model structure that any price instrument $(Z)$ that is correlated with $p_j$ will also be correlated with $\boldsymbol{\eta}(\boldsymbol{\delta_t})$, since $\delta$ is a function of $p_j$. This correlation results in $Cov\left(Z,\boldsymbol{\eta}(\boldsymbol{\delta_t})\right)\neq 0$ in small samples and $plim_{N\to\infty}N^{-1}\left(Z^T\boldsymbol{\eta}(\boldsymbol{\delta_t})\right)\neq 0$ in large sample, which leads to $\hat{\alpha}^p_{IVS}$ being a biased and inconsistent estimate of $\alpha^p$. □

The above proof illustrates that the bias and inconsistency associated with the IVS estimator is a function of the relationship between $Z$ and the approximation error, $\boldsymbol{\eta}(\boldsymbol{\delta_t})$ when the true data generating process has consumers tracking each state variable. Our research is the first to theoretically highlight the small and large sample properties of the IVS estimator. While we illustrate the theoretical properties of the IVS estimator, we also want to understand their practical accuracy across different plausible scenarios. It may be possible that while the IVS estimator is biased and inconsistent, its bias may be low in practice. The above proposition motivates the use of Monte Carlo simulations to illustrate the degree of bias associated with the IVS method under a variety of realistic parameterizations and conditions. Finally, we should note the obvious in that as $\beta\to 0$ any bias generated by the approximation method also tends toward zero.

It is important to highlight that Andrews and Armstrong (2017) show that in the case of a single instrument, their estimator for the price parameter $\alpha^p$ is unbiased if the sign of $\gamma$ is known and the errors of the first and second stage equations are normally distributed. They further show the IV estimator behaves equivalent to their unbiased estimator when the first stage instrument is strong. Note, the sign of the relationship between price and the instrument, e.g. a BLP instrument, is a fairly weak assumption in our context. Thus, they show that an instrumental variable estimate of $\alpha^p$ is unbiased when $Cov(\mathbf{Z},\boldsymbol{\xi})=0$, $Cov(\mathbf{Z},\boldsymbol{p})\neq 0$, the sign of $\gamma$ is known, the instrument is strong, and the reduced form errors of the first and second stage equations $(\xi,\nu)$ are normally distributed. This finding is important as it enables our

Monte Carlo simulations in Section 5 to characterize the bias attributed to the IVS approximation error.

### 4.1 When might IVS not be biased / inconsistent?

It is worthwhile to understand the boundary conditions to examine the source of the error in IVS. We consider two cases, first when we might have a measurement error and second when IVS is a behavioral assumption.

1. **Measurement Error:** First, suppose that the approximation error $\eta(\delta_t)$ is truly a measurement error, or if the approximation error merely added some noise to the expected value function. However, we note that the approximation error is due to the nonlinear mapping from a multidimensional state-space to a single-dimensional state space. Therefore, in general it is likely to depend on the state variables, including product characteristics and prices.
2. **Behavioral Assumption:** Second, while the proposition does illustrate an associated bias and inconsistency of the IVS estimator, it is important to highlight that our conclusion is based on the true data generation process consisting of consumers forming expectations over all relevant state variables. In practice, the econometrician is not informed of the *true underlying model.* For example, consumers might look to a sufficient statistic to track as an aggregate measure of the state when the number of products and state space is large. Thus, in order to unconditionally make a claim that the IVS estimator is inferior to the full solution method, one needs a different analysis of these two settings with a test that can determine which model of dynamic product adoption is more likely to be valid.[6] Therefore, it is possible that the IVS estimation procedure may perform better than the full solution if consumers do in fact form future expectations employing the inclusive value statistic.

## 5 Monte Carlo model and results

The data generating process follows the above full state variable model. For simplicity, we only include price and not product characteristics. The number of products varies as $J \in \{2, 3, 4, 5\}$ . We parameterize the DGP so that prices are decreasing over time, consistent with the durable goods adoption market setting, and that the generated long term elasticity estimates are realistic (fall with in range of -1 and -3). The state variables in our Monte Carlo model are $(\mathbf{p},\boldsymbol{\xi})$ with parameters set at $\alpha_j = \alpha = 0.5$. We consider two price parameterizations where $\alpha^p \in \{-0.2, -0.3\}$ with two values of the consumers' discount factor ($\beta = 0.80$ and $\beta = 0.95$). Note, we vary $\beta$ to illustrate numerically, that the bias shown in the proof above does decreases when $\beta \to 0$. We also allow the price processes to vary. In the first set of simulations

[6] We thank the co-Editor for pushing us to clearly state under what conditions and assumptions these theoretical results hold.

we restrict these processes to be identical across products. In the second set, we relax this restriction and allow for differing price processes for each product.

We characterize the performance of the IVS estimator compared to that of a full solution method. Note that this might not always be possible since in many cases, the full solution method is computationally intractable. We present the results from a benchmark full-solution method in order to identify the relative applicability of the IVS estimator over its full solution counterpart.

### 5.1 What parameterizations should be chosen?

When generating our simulated data, it is important that the parameterization generates realistic data. The measure we use to determine the realistic nature of the data is long run elasticity. Many empirical IO and marketing papers using data from different industries generate own price elasticities within the range of [1 to 3] (see for example Nair (2007)). For our simulations, the above parameterizations generate long run elasticities in the above range. The actual elasticity depends upon several factors including the number of products in the market place, the discount factor and the price coefficient.

In addition, we employ long run own-price elasticities and profits to inform the reader of the difference between the data generating process and the results of the estimation methodologies, as both of these measures are economically meaningful to researchers. Specifically, our measure of long run own-price elasticity is the % change in total quantity for good $j$ for the first 25 periods resulting from a 1% permanent decrease in the price of good $j$.[7] While we are are agnostic to the time interval of the 25 periods, they can represent weeks, months or even quarters. At the quarterly interval an annualized discount factor corresponding to $\beta = 0.95$ is slightly larger than what was found in Dubé et al. (2014) of 0.8. We determine this measure for each good and average over the number of products as follows:

$$E_p = 100 \times \frac{1}{J} \sum_{j=1}^{J} \left[ \frac{\sum_{t=1}^{t=25} (Q_{j,t}(p'_{j,t}) - Q_{j,t}(p_{j,t}))}{\sum_{t=1}^{t=25} Q_{j,t}(p_{j,t})} \right]$$

The profit measure is computed as the sum of discounted period profits and computed based on prices, marginal costs and sales in each period. We then determine the percent change in profit from a 1% decrease in price, assuming an initial market size of $I = 10,000,000$ consumers and a discount factor of $\Upsilon = 0.975$ for the firm. The “profit elasticity” is defined as:

$$E_\pi = 100 \times \frac{1}{J} \sum_{j=1}^{J} \left[ \frac{\sum_{t=1}^{t=25} \Upsilon^t \left[ \left( p'_{j,t} - MC_{j,t} \right) Q_{j,t}(p'_{j,t}) - \left( p_{j,t} - MC_{j,t} \right) Q_{j,t}(p_{j,t}) \right]}{\sum_{t=1}^{t=25} \Upsilon^t \left( p_{j,t} - MC_{j,t} \right) Q_{j,t}(p_{j,t})} \right])$$

[7]In implementing the price change, we use changes from a generated price and marginal cost trajectory, and retain the same error terms under the changed prices.

### 5.2 Price process: Identical across products

In this first set of simulation results we assume the data generating process is identical for all products, allowing us to focus on investigating how the estimation method performs as $\bar{\alpha}^p$, $J$ and $\beta$ vary. The price process for all simulations is a function of marginal cost and an error term $\nu_{j,t} \sim N\left(0, \sigma_\nu^2\right)$ with $\sigma_\nu = 0.25$ that is correlated with $\xi_{j,t}$. Such a formulation is motivated by the price endogeneity problem researchers face when employing aggregate data, where firms can observe $\xi_{j,t}$ and then set prices optimally. We use a reduced form model to specify this dependence:

$$p_{j,t} = \theta_1 + MC_{j,t} + \nu_{j,t}$$

where $\theta_1 = 3$ and $MC_{j,t}$, is uncorrelated with the current period unobserved product characteristic (structural error) $\xi_{j,t} \sim N\left(0, \sigma_\xi^2\right)$ iid across $J$ and $T$, with $\sigma_\xi = 0.005$. Finally, the initial marginal cost for each of the $J$ products take the value $MC_{j,0} = 9$. Consumers are homogeneous in preferences, $\sigma_p = 0$.

Given that current prices are correlated with $\xi_{j,t}$, it is required that we have an excluded instrument correlated with $p_{j,t}$, but is uncorrelated with $\xi_{j,t}$. Absent product characteristics that could help generate BLP-type instruments, the natural instrument here is the product's marginal cost, which has a decaying trajectory, consistent with a durable goods model:

$$MC_{j,t} = 0.35 + 0.925 MC_{j,t-1} + \kappa_{j,t}$$

where $\kappa_{j,t} \sim N\left(0, \sigma_\kappa^2\right)$ and $\sigma_\kappa = 0.25$. Note that $\kappa_{j,t}$ is uncorrelated with price or the structural error.

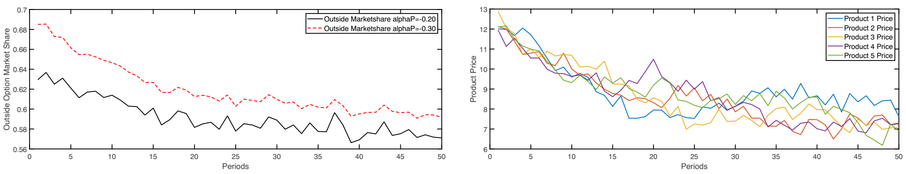
With three random variables associated with the data generating process, we summarize the distributional properties of $\xi_{j,t}$, $\nu_{j,t}$ and $\kappa_{j,t}$:

$$\begin{pmatrix} \xi \\ \nu \\ \kappa \end{pmatrix} \sim N\left(0, \sum\right) \text{ with } \sum = \begin{pmatrix} \sigma_\xi^2 & \rho\sigma_\xi\sigma_\nu & 0 \\ \rho\sigma_\xi\sigma_\nu & \sigma_\nu^2 & 0 \\ 0 & 0 & \sigma_\kappa^2 \end{pmatrix} \text{ and } \rho = 1$$

This specification is useful in generating sizeable correlation between the unobserved structural error and price, as is typically the case with aggregate sales data.

In Fig. 1, we present the corresponding market share for the outside option (for $J = 5$ products) for the two different price parameterizations along with the price paths for each of the five products (for the simulation NS=1). As is evident, the outside option declines over time, and its market share is smallest under the parameterization of $\bar{\alpha}^p = -0.2$ and largest with $\bar{\alpha}^p = -0.3$. Furthermore, given the underlying state transition process is identical across all products, the price paths for all five products are similar, with differences only driven by the random noise in the marginal cost and price processes.

Table 1 below presents the estimation results of the IVS solution and the full solution when the state transition variables follow an identical transition process. In Appendix A, we outline the steps taken to estimate the IVS model. Table 2 includes




(a) DGP Outside Market Share (b) DGP Product Prices

**Fig. 1** Outside market share and prices

the % difference in the long run elasticity and profit from the data generating process values for each estimation method. Within Table 1, the most notable fact is the difference in parameter estimates employing the IVS and full solution methods. The IVS method demonstrates a larger bias and less precision for all price parameters, number of products and discount factor. That said, the estimates do improve as the number of products $J$ increases, but at a diminishing rate. For example, with the discount factor set at $\beta = 0.95$ and $\bar{\alpha}^p = -0.3$ the marginal improvement from adding one more product changes from 0.09 ($j = 2$ to $j = 3$) to 0.02 ($j = 3$ to $j = 4$) and 0.01 for ($j = 4$ to $j = 5$). This same pattern is evident with $\bar{\alpha}^p = -0.2$ and with $\beta = 0.80$.[8]

In order to translate the illustrated bias into economically meaningful terms, we characterize the long term *own-price elasticities* from a permanent 1% decline in price as well as the percent change in *profit* from this permanent price change. These measures are determined using the first 25 periods of data. Like in the case of the parameter estimates, the full solution model dominates the IVS estimator when it comes to recovering each statistic. The IVS estimator is also found to underestimate the model's long term own-price elasticities whereas profits are over estimated. Additionally, note that the own-price elasticity measures do improve as $J$ increases, but the profit measure does not. It is important to recognize why. With the own-price elasticity, the model only tracks the total number of units sold over the first 25 periods, regardless of when they are sold. However, with respect to profit, what matters is not simply quantity but the timing of which those quantities are sold, given prices decline with time and future periods are discounted. Thus, our Monte Carlo simulations illustrate that the IVS procedure is better at recovering own-price elasticities than shifts in profits when price sensitivities are large ($\alpha_p = -.3$ vs $\alpha_p = -.2$). and/or when the discount factor is small ($\beta = .8$ vs $\beta = .95$).

In summary, we observe that the full solution estimates are almost identical to the true parameter values as expected. The parameter estimates from the IVS method, on

[8] We do generate data associated with J=8 but we do not report these results given the computational time that is required to form elasticity estimates given the DGP. With 8 products we must determine new equilibrium beliefs for each measure of own-price elasticity for each NS simulation run. We have estimated the model using this data for both estimators and have found that the IVS parameter estimates exhibit more bias than the setting of J=5 and in some cases J=2. Thus, it appears the *improvement* of the IVS estimator is nonlinear and is eliminated when J is large (J=8).

**Table 1** Monte Carlo simulation results

| | | $\beta = 0.95$ | | | | $\beta = 0.80$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| True Values | | i: IVS | | ii: Full Solution | | i: IVS | | iii: Full Solution | |
| | | $\alpha$ | $\bar{\alpha}^p$ | $\alpha$ | $\bar{\alpha}^p$ | $\alpha$ | $\bar{\alpha}^p$ | $\alpha$ | $\bar{\alpha}^p$ |
| $\alpha = 0.5, \bar{\alpha}^p = -0.2$ | T=100; J=2 | 0.456 (0.006) | −0.104 (0.012) | 0.500 (0.004) | −0.201 (0.009) | 0.471 (0.010) | −0.182 (0.006) | 0.500 (0.004) | −0.200 (0.002) |
| | T=100; J=3 | 0.461 (0.005) | −0.117 (0.010) | 0.500 (0.003) | −0.201 (0.007) | 0.473 (0.008) | −0.183 (0.005) | 0.500 (0.003) | −0.200 (0.002) |
| | T=100; J=4 | 0.464 (0.005) | −0.124 (0.010) | 0.500 (0.002) | −0.200 (0.005) | 0.473(0.007) | −0.183 (0.004) | 0.500 (0.003) | −0.200 (0.002) |
| | T=100; J=5 | 0.467 (0.005) | −0.130 (0.010) | 0.500 (0.002) | −0.200 (0.005) | 0.473 (0.006) | −0.183 (0.003) | 0.500 (0.003) | −0.200 (0.002) |
| $\alpha = 0.5, \bar{\alpha}^p = -0.3$ | T=100; J=2 | 0.471 (0.011) | −0.234 (0.023) | 0.500 (0.004) | −0.301 (0.009) | 0.460 (0.014) | −0.276 (0.008) | 0.500 (0.004) | −0.300 (0.002) |
| | T=100; J=3 | 0.474 (0.009) | −0.243 (0.018) | 0.500 (0.003) | −0.301 (0.007) | 0.463 (0.012) | −0.278 (0.007) | 0.500 (0.003) | −0.300 (0.002) |
| | T=100; J=4 | 0.476 (0.008) | −0.245 (0.016) | 0.500 (0.002) | −0.300 (0.005) | 0.466 (0.011) | −0.279 (0.006) | 0.500 (0.003) | −0.300 (0.002) |
| | T=100; J=5 | 0.477 (0.008) | −0.246 (0.015) | 0.500 (0.002) | −0.300 (0.005) | 0.467 (0.009) | −0.279 (0.005) | 0.500 (0.003) | −0.300 (0.002) |

This table reports the parameter estimates of several monte carlo exercises of estimating a dynamic discrete choice demand model. Exercises vary the discount factor ($\beta$) from 0.95 to 0.80 and the preference parameter for price ($\bar{\alpha}^p$) from −0.2 to −0.3. Estimation assumes each product is a terminal choice and that once a consumer purchases, (s)he exits the market. The DPG assumes consumers track all state variables in forming expectations about the future. There is no differentiation in the transition processes of state variables across products. The IVS method assumes the econometrician estimates the demand model where consumers follow only the inclusive value statistic in forming expectations whereas the Full Solution method the econometrician assumes that consumer track all state variables

Mean and standard deviation for 250 simulations

**Table 2** Long term elasticities: Identical transition

| True Values | | $\beta = 0.95$ | | | | | | $\beta = 0.80$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | DGP | | i: IVS | | iii: Full Solution | | DGP | | i: IVS | | iii: Full Solution | |
| | | $E_p$ | $E_\pi$ | $\% \Delta E_p$ | $\% \Delta E_\pi$ | $\% \Delta E_p$ | $\% \Delta E_\pi$ | $E_p$ | $E_\pi$ | $\% \Delta E_p$ | $\% \Delta E_\pi$ | $\% \Delta E_p$ | $\% \Delta E_\pi$ |
| $\alpha = 0.5, \bar{\alpha}^p = -0.2$ | T=100; J=2 | 1.14 | −2.70 | −47.68 | 19.70 | 0.74 | 0.93 | 1.13 | −2.68 | −8.86 | 3.71 | 0.07 | 3.20 |
| | T=100; J=3 | 1.53 | −2.33 | −41.55 | 26.48 | 0.51 | −0.44 | 1.53 | −2.34 | −8.39 | 5.36 | 0.06 | 1.69 |
| | T=100; J=4 | 1.73 | −2.15 | −38.11 | 29.79 | −0.02 | −0.56 | 1.73 | −2.16 | −8.48 | 6.61 | 0.04 | 0.90 |
| | T=100; J=5 | 1.85 | −2.05 | −35.26 | 30.70 | 0.36 | −0.78 | 1.85 | −2.07 | −8.49 | 7.39 | 0.06 | 0.56 |
| $\alpha = 0.5, \bar{\alpha}^p = -0.3$ | T=100; J=2 | 1.69 | −2.12 | −21.64 | 16.98 | 0.70 | −1.19 | 1.65 | −2.03 | −7.97 | 6.57 | −0.55 | 11.90 |
| | T=100; J=3 | 2.28 | −1.57 | −19.10 | 26.93 | 0.43 | −2.27 | 2.24 | −1.53 | −7.23 | 10.41 | −0.17 | 8.55 |
| | T=100; J=4 | 2.58 | −1.29 | −18.36 | 35.59 | 0.01 | −1.47 | 2.54 | −1.27 | −7.02 | 13.70 | −0.15 | 6.10 |
| | T=100; J=5 | 2.78 | −1.14 | −17.95 | 41.93 | 0.26 | −1.98 | 2.73 | −1.14 | −6.79 | 15.87 | −0.04 | 4.87 |

This table reports the long term elasticity results of several monte carlo exercises. Exercises vary the discount factor ($\beta$) from 0.95 to 0.80 and the preference parameter for price ($\bar{\alpha}^p$) from −0.2 to −0.3. Estimation assumes each product is a terminal choice and that once a consumer purchases, (s)he exits the market. The DPG assumes consumers track all state variables in forming expectations about the future. There is no differentiation in the transition processes of state variables across products. The IVS method assumes the econometrician estimates the demand model where consumers follow only the inclusive value statistic in forming expectations whereas the Full Solution method the econometrician assumes that consumer track all state variables. $E_p$ and $E_\pi$ are the average Long term price and profit elasticities associated with a 1% change in price over all $J$ products. $\%\Delta$ correspond to the percent change from the observed DGP elasticity measures

Mean and standard deviation for 250 simulations

the other hand are significantly different from the true values. Specifically, we find the following:

1. The price coefficient recovered through the IVS method is biased towards zero, thus implying that consumers are less price sensitive than they actually are. We note that this has a strong impact on elasticities, that we examine in detail below. The constant term also seems biased towards zero, although in our parametrization the error seems relatively smaller for this term compared to the price coefficient.
2. The standard errors recovered from IVS are almost always higher than for the full solution model. The confidence interval for the IVS parameter estimates in most cases does not include the true parameter values.
3. As the number of products in the market $J$ increases from 2 to 5, the error in both the price coefficient and for the constant terms diminishes, although the rate of decrease is small.
4. When consumers are more price sensitive in reality, it increases the accuracy of the estimates from the IVS method. The full solution method is accurate for both lower and higher price sensitivities.
5. For a lower discount factor $\beta = 0.8$ relative to $\beta = 0.95$, the error in estimating the price coefficient is significantly diminished. This finding is consistent with the theory, where the bias has $\beta$ as a proportional term.

From Table 2, we we make the following observations about the results of elasticities and profit changes resulting from a small change in prices:

1. With a high discount factor of $\beta = 0.95$, the elasticities derived from the IVS model estimates are between 35% and 47% different from the true elasticities. The profit changes with respect to the price change also vary in the range of 20-30%.
2. With higher price sensitivity, the error in the price elasticities is lower, but the profit elasticities can be higher. Thus, the range of parametrizations that IVS is suitable for might depend on what the estimates are used for (profit or elasticity).
3. Again, similar to the parameter estimates, we find that with a lower discount factor, the error with IVS is not as high.

The elasticities and profit impact are directionally similar, however, the quantitative impact is different. Observe that the elasticity measure is only characterized by the aggregate quantity, whereas the profit measure depends on *when* products are purchased, and how they are discounted. If price changes affect the intertemporal purchase patterns, that will have more of an impact on profits but not on elasticities.

Above, we have detailed the main results when all products follow the same price process. Next, we examine the parameter estimates and elasticities and profit impact when the products follow different price processes respectively in Tables 3 and 4. We find very similar patterns of results as we had reported above, with some quantitative differences. The error variation is directionally the same with respect to the price sensitivity, discount factor and the number of products in the market when we have different price processes across products. Again, the full solution method does not suffer from these inaccuracies, but at the cost of higher computational complexity.

**Table 3** Monte Carlo simulation results: Different price process

| True Values | | DGP: One Market: $\beta = 0.95$ | | | | DGP: One Market: $\beta = 0.80$ | | | |
|---|---|---|---|---|---|---|---|---|---|
| | | i: IVS | | ii: Full Solution | | i: IVS | | iii: Full Solution | |
| | | $\alpha$ | $\bar{\alpha}^p$ | $\alpha$ | $\bar{\alpha}^p$ | $\alpha$ | $\bar{\alpha}^p$ | $\alpha$ | $\bar{\alpha}^p$ |
| $\alpha = 0.5, \bar{\alpha}^p = -0.2$ | T=100; J=2 | 0.459 (0.009) | −0.121 (0.020) | 0.500 (0.001) | −0.200 (0.003) | 0.477 (0.010) | −0.187 (0.005) | 0.500 (0.001) | −0.200 (0.001) |
| | T=100; J=3 | 0.466 (0.007) | −0.135 (0.015) | 0.500 (0.001) | −0.200 (0.003) | 0.480 (0.008) | −0.189 (0.005) | 0.500 (0.001) | −0.200 (0.001) |
| | T=100; J=4 | 0.471 (0.006) | −0.143 (0.012) | 0.500 (0.002) | −0.200 (0.005) | 0.481(0.007) | −0.189 (0.004) | 0.500 (0.001) | −0.200 (0.001) |
| | T=100; J=5 | 0.475 (0.006) | −0.150 (0.012) | 0.500 (0.002) | −0.200 (0.005) | 0.482 (0.006) | −0.189 (0.003) | 0.500 (0.001) | −0.200 (0.001) |
| $\alpha = 0.5, \bar{\alpha}^p = -0.3$ | T=100; J=2 | 0.471 (0.012) | −0.245 (0.024) | 0.500 (0.001) | −0.300 (0.003) | 0.460 (0.014) | −0.276 (0.008) | 0.500 (0.001) | −0.300 (0.001) |
| | T=100; J=3 | 0.479 (0.009) | −0.259 (0.018) | 0.500 (0.002) | −0.300 (0.003) | 0.463 (0.012) | −0.278 (0.007) | 0.500 (0.001) | −0.300 (0.001) |
| | T=100; J=4 | 0.482 (0.007) | −0.263 (0.015) | 0.500 (0.002) | −0.300 (0.003) | 0.466 (0.011) | −0.279 (0.006) | 0.500 (0.001) | −0.300 (0.001) |
| | T=100; J=5 | 0.484 (0.007) | −0.265 (0.013) | 0.500 (0.002) | −0.300 (0.003) | 0.467 (0.009) | −0.279 (0.005) | 0.500 (0.001) | −0.300 (0.001) |

This table reports the parameter estimates of several monte carlo exercises of estimating a dynamic discrete choice demand model. Exercises vary the discount factor ($\beta$) from 0.95 to 0.80 and the preference parameter for price ($\bar{\alpha}^p$) from $-0.2$ to $-0.3$. Estimation assumes each product is a terminal choice and that once a consumer purchases, (s)he exits the market. The DPG assumes consumers track all state variables in forming expectations about the future. In this set of monte carlos there is measurable differentiation in the transition processes of state variables across products. The IVS method assumes the econometrician estimates the demand model where consumers follow only the inclusive value statistic in forming expectations whereas the full solution method the econmetrician assumes that consumer track all state variables

Mean and standard deviation for 250 simulations

**Table 4** Long term elasticities: Different price process

| True Values | | $\beta = 0.95$ | | | | | | $\beta = 0.80$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | DGP | | i: IVS | | iii: Full Solution | | DGP | | i: IVS | | iii: Full Solution | |
| | | $E_p$ | $E_\pi$ | $\% \Delta E_p$ | $\% \Delta E_\pi$ | $\% \Delta E_p$ | $\% \Delta E_\pi$ | $E_p$ | $E_\pi$ | $\% \Delta E_p$ | $\% \Delta E_\pi$ | $\% \Delta E_p$ | $\% \Delta E_\pi$ |
| $\alpha = 0.5, \bar{\alpha}^p = -0.2$ | T=100; J=2 | 1.20 | −2.83 | −39.23 | 16.15 | 0.94 | −2.74 | 1.20 | −2.82 | −6.09 | 2.55 | 0.21 | −0.60 |
| | T=100; J=3 | 1.58 | −2.41 | −32.51 | 20.71 | 1.01 | −5.31 | 1.58 | −2.41 | −5.68 | 3.62 | 0.28 | −2.35 |
| | T=100; J=4 | 1.77 | −2.19 | −28.46 | 22.26 | 0.63 | −6.79 | 1.77 | −2.12 | −5.68 | 4.42 | 0.22 | −2.12 |
| | T=100; J=5 | 1.86 | −2.06 | −25.18 | 21.96 | 0.89 | −7.20 | 1.86 | −2.08 | −5.53 | 4.79 | 0.29 | −3.36 |
| $\alpha = 0.5, \bar{\alpha}^p = -0.3$ | T=100; J=2 | 1.79 | −2.24 | −18.07 | 14.17 | 2.36 | −9.21 | 1.65 | −2.03 | −7.97 | 6.57 | 0.35 | −1.45 |
| | T=100; J=3 | 2.36 | −1.63 | −13.71 | 19.36 | 2.05 | −11.03 | 2.24 | −1.53 | −7.23 | 10.41 | 0.70 | −7.78 |
| | T=100; J=4 | 2.63 | −1.31 | −12.38 | 24.04 | 1.67 | −16.07 | 2.54 | −1.27 | −7.02 | 13.70 | 0.68 | −11.06 |
| | T=100; J=5 | 2.77 | −1.14 | −11.84 | 27.81 | 1.96 | −14.44 | 2.73 | −1.14 | −6.79 | 15.87 | 0.76 | −13.06 |

This table reports the long term elasticity results of several monte carlo exercises. Exercises vary the discount factor ($\beta$) from 0.95 to 0.80 and the preference parameter for price ($\bar{\alpha}^p$) from $-0.2$ to $-0.3$. Estimation assumes each product is a terminal choice and that once a consumer purchases, (s)he exits the market. The DPG assumes consumers track all state variables in forming expectations about the future. There is measurable differentiation in the transition processes of state variables across products. The IVS method assumes the econometrician estimates the demand model where consumers follow only the inclusive value statistic in forming expectations whereas the full solution method the econmetrician assumes that consumer tracks all state variables. $E_p$ and $E_\pi$ are the average Long term price and profit elasticities associated with a 1% change in price over all $J$ products. $\%\Delta$ correspond to the percent change from teh observed DGP elasticity measures

Mean and standard deviation for 250 simulations

### 5.3 Price process: Different across products

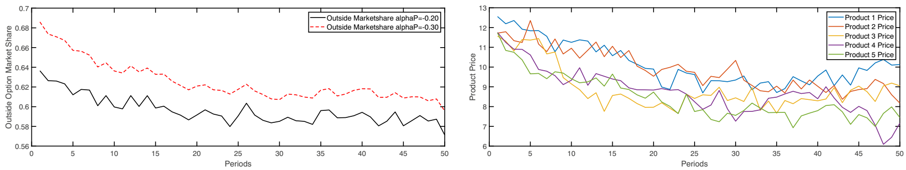
The next set of Monte Carlos simply expands on the simulations above by creating further heterogeneity in the price process by increasing the range of $\rho_j$ from [0.925] to [0.965, 0.895]. The marginal cost constant also adjusts from 0.35 to a range of [0.21, 0.49]. Everything else remains identical to the above previously discussed price process.

$$\begin{aligned} MC_{j,t} &= \theta_{mc,j} + \rho_j MC_{j,t-1} + \kappa_{j,t} \\ \rho_j &= [0.965;0.94;0.925;0.91;0.895] \\ \theta_{mc,j} &= [0.21;\ 0.28;\ 0.35;\ 0.42;\ 0.49]. \end{aligned}$$

Like above, we provide Fig. 2 to illustrate the outside market share under each price parameterization and in sub-figure two the greater dispersion of price across all five products than presented above. The results in Table 3 are very much similar to the results in Table 1– the IVS approximation has the potential for substantial bias under all parameterizations relative to the full solution method and that its performance also improves as the discount factor declines and/or when the consumers price sensitivity is large ($\alpha_p = -.3$). There is also a similar trend with respect to the bias associated with the price parameter estimate; it too improves as the number of products $J$ increases, but again at a diminishing rate.

From the presented Monte Carlo simulations in Tables 1 and 3, we determine the performance of the IVS estimator varies depending upon the setting. Given that we have discussed the results of the IVS estimator under the two price processes separately, we limit our discussion to analyzing the similarities and differences across these two settings. It is clear that when the underlying data generating process of the relevant state variables becomes more differentiated across products, estimates of the parameters improve, regardless of the consumer's price sensitivity. The percent change from the DGP elasticities also improves as the price process across products becomes more differentiated and as J increases.

However, these improvements do not compete with the results of the full solution method, where little to no bias is observed in parameter estimates and in the estimates of long term (quantity) elasticities. With respect to the long term change in profits




(a) DGP Outside Market Share

(b) DGP Product Prices

**Fig. 2** Outside market share and prices from a permanent 1% reduction in price, the full solution does exhibit some deviation from the DGP value but this deviation is smaller than the what is found employing the IVS method. Finally, the IVS method appears to perform better in recovering parameter estimates and in matching the long term elasticies when the consumer price sensitivity is large ($\alpha_p = -.3$ vs $\alpha_p = -.2$) and/or when the consumer discount factor is small ($\beta = .8$ vs $\beta = .95$), regardless of the price process. The latter result is unsurprising given Proposition 1– the bias associated with the parameter estimates decreases as $\beta \rightarrow 0$.[9]

Additionally, the Monte Carlo results for both price processes illustrate that the IVS method under estimates the model's own-price elasticities and over estimates the change in profits as prices change. This finding highlights the importance of correctly characterizing how consumers form expectations. If consumers form expectations by tracking each individual state variable, the result of the model misspecification by employing an IVS estimator can be quite large with deviations of own-price elasticities upward of -50% in our Monte Carlo settings. Moreover, with the IVS price parameter estimate biased toward zero, the associated cross-price elasticities will be biased toward zero as well. This implies a less competitive environment than what is true. The downward bias of a product's own-price and cross-price elasticities can have important ramifications for antitrust/merger analysis as regulators would incorrectly assume that a firm has to much market power. Thus, when employing dynamic demand models to understand important policy interventions, its is vital the researcher empirically supports his/her decision about the underlying consumer belief structure. One such method is for regulators or policy analyst to directly survey consumers about their beliefs as is discussed in Manski (2004).

## 6 Discussion and conclusion

Dynamic discrete choice models are typically computationally intractable without using approximation methods (Aguirregabiria and Mira 2010). The inclusive value is one such approximation that makes estimation tractable. In our examination, we show that the estimator generates bias in the parameter estimates when compared to the full solution method. We also show that it performs better when the discount factor is small ($\beta = .8$ vs $\beta = .95$) and/or when the price sensitivity of the consumer is large ($\alpha_p = -.3$ vs $\alpha_p = -.2$). Additionally, predictions of economic quantities of interest (long-run demand and profits) indicate potential for substantial bias. Our analysis does have one important limitation and that it excludes the inclusion of unobservable consumer heterogeneity with respect to the price. We have experimented with allowing for unobserved heterogeneity, but found that without the right type of data variation the estimator performs poorly. In order to estimate this parameter we

[9] In Appendix A, we present the results of our analysis of a short-term temporary price increase.

require significant variation in the data, which is computational challenging to generate given the need for variation across multiple markets, time periods and products. Moreover, we have no reason to expect that the introduction of heterogeneity makes the inclusive value estimator *less biased*.

Many practical settings of interest feature a large number of products. For example, Gowrisankaran and Rysman (2012) consider the market for digital camcorders with $J = 383$ products, and similarly the automobile market modeled in Berry et al. (1995) had hundreds of models as well. We note that in such settings, it might not even be possible to check whether the IVS approach is appropriate, given that the full model might be quite intractable. Thus, an alternative approach might be to model the primary observable product characteristic or a small subset of characteristics as the state variables of interest (Gordon 2009; Song and Chintagunta 2003), However, this approach has the disadvantage that crucial unobservable time-varying product characteristics that determine consumer choices could be left out of the model. In some special cases, it might be possible to avoid making the inclusive value approximation by using an estimation approach that sidesteps the need to compute a value function, similar to Bayer et al. (2016), who examine the demand for housing with an individual model or Chou et al. (2019) who estimate dynamic discrete choice demand models using aggregate data. Broadly, it points to the need for further research into dynamic discrete models to develop better approximations or alternative methods to address this challenging issue.

The concern regarding approximations used in dynamic discrete choice models are not limited to those of the inclusive value kind. Rather, there are other widely used approximations in dynamic discrete choice models, particularly with individual level data. The impact of these approximations has been an under-explored area of research. They are important because they can result in significant bias not only in the obtained parameter estimates, but in counterfactual objects of interest, including elasticities and welfare measures.

Our work also produces additional questions that are worth building upon in further research. First, the approximations errors identified here are conceptually present even when IVS is used with individual-level data in stockpiling models, e.g. Hendel and Nevo (2006). It would be useful to examine whether there are specific features of problem settings that might make such a method more accurate, e.g. having a relatively low number of products. Second, if we are able to identify the expected asymptotic error from IVS, it might be possible to develop a "bias correction" for it. Even if that proves challenging, it might be useful to explore whether we might be able to bound the errors identified in the paper. This is likely to depend on the specifics of the model, but some classes of problems (e.g. with exit choices similar to our setting of interest) might have simplifications to allow such bounding.

## Appendix A: Short term elasticity

In addition to estimates of a long run own-price elasticities and profits to inform the reader of the difference between the data generating process and the results of the estimation methodologies, we also present short-term elasticities. Specifically, this short-term own-price elasticity is the % change in total quantity for good $j$ for the first $\tau_{\text{short}} = 4$ periods resulting from a 1% temporary decrease in the price of good $j$ in period 1. We determine this measure for each good and average over the number of products as done above with the long-term elasticity measures.The profit measure is computed as the sum of discounted period profits and computed based on prices, marginal costs and sales in each period. We then determine the percent change in profit from a 1% temporary decrease in price in period 1, assuming an initial market size of 10,000,000 consumers and a discount factor of $\beta_j = 0.975$ for the firm.

We present the results of a short-term temporary price change in Tables 5 and 6 for the setting of identical price transitions and heterogeneous transitions, respectively. When analyzes the results we find the results from the own-price elasticity differs from the long term elasticity above. Specifically, the tables below indicate that % change from the short-term own-price DGP elasticity becomes more negative as J increases whereas the long-term own-price elasticity improves as J increases. Moreover, it appears that with respect to both elasticity measures that the IVS is competitive with the full solution when J=2 or J=3.

## Appendix B: Computational details

We use the following computational algorithm to estimate the model parameters. We employ a GMM procedure using mathematical programming with equilibrium constraints (MPEC). Model parameters are $\theta = (\bar{\alpha}^p, \alpha)$. Let $W$ be the GMM weighting matrix. The constrained optimization formulation is

$$\min_{\theta,\xi} \left[\xi' \mathbf{ZWZ}' \xi\right],$$
$$st : \hat{s}_{jt}(\xi, \theta) = S_{jt}$$

with the market share equations imposed as constraints to the optimization problem.

### Overall procedure

1. Given a guess of $\theta = (\bar{\alpha}^p, \alpha)$ and $\xi_{jt}$ determine the simulated market share for each product in each time period.
2. With the same guess of $\theta = (\bar{\alpha}^p, \alpha)$ and $\xi_{jt}$ compute the GMM objective function defined in the equation above.
3. Search over $\theta = (\bar{\alpha}^p, \alpha)$ and $\xi_{jt}$to minimize the objective function given the constraint that the observed market share equals the simulated share.

### Formation of the market share constraint

1. Given a guess of $\theta = (\bar{\alpha}^p, \alpha)$ and $\xi_{jt}$ formulate $f_{k,t}\left(x_t^c, \xi_t\right)$ for each product $k$, and for each period $t$.

**Table 5** Short term elasticities: Identical transition

| | | DGP: One Market: $\beta = 0.95$ | | | | | | DGP: One Market: $\beta = 0.80$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | DGP | | i: IVS | | iii: Full Solution | | DGP | | i: IVS | | iii: Full Solution | |
| | | $E_p^s$ | $E_\pi^s$ | $\% \Delta E_p^s$ | $\% \Delta E_\pi^s$ | $\% \Delta E_p^s$ | $\% \Delta E_\pi^s$ | $E_p^s$ | $E_\pi^s$ | $\% \Delta E_p^s$ | $\% \Delta E_\pi^s$ | $\% \Delta E_p^s$ | $\% \Delta E_\pi^s$ |
| $\bar{\alpha}^p = -0.2$ | T=50; J=2 | 0.52 | −1.19 | −33.22 | 14.99 | −13.56 | 9.32 | 0.52 | −1.09 | 11.47 | −5.76 | −12.22 | 11.13 |
| | T=50; J=3 | 0.70 | −1.05 | −35.22 | 23.23 | −6.68 | 7.24 | 0.69 | −1.02 | −0.81 | −0.68 | 11.05 | 10.46 |
| | T=50; J=4 | 0.79 | −0.98 | −34.29 | 27.58 | 9.48 | −6.32 | 0.80 | −0.98 | −3.78 | 2.83 | −0.74 | −1.99 |
| | T=50; J=5 | 0.84 | −0.95 | −34.28 | 30.72 | −2.93 | 3.56 | 0.87 | −0.98 | −6.26 | 5.56 | 3.00 | −3.90 |
| $\bar{\alpha}^p = -0.3$ | T=50; J=2 | 0.74 | −0.83 | 3.06 | −2.83 | −2.90 | 4.78 | 0.72 | −0.60 | 13.37 | −16.71 | 2.93 | 8.44 |
| | T=50; J=3 | 0.97 | −0.63 | −8.46 | 13.09 | −12.40 | 21.56 | 0.90 | −0.50 | 3.17 | −5.89 | −4.76 | 2.12 |
| | T=50; J=4 | 1.09 | −0.52 | −13.20 | 26.99 | 8.73 | −17.39 | 1.02 | −0.45 | −2.71 | 5.50 | −2.01 | 8.23 |
| | T=50; J=5 | 1.17 | −0.47 | −16.70 | 41.27 | −2.73 | 3.68 | 1.10 | −0.42 | −6.56 | 16.66 | −0.27 | −2.27 |

This table reports the short term elasticity results of several monte carlo exercises. Exercises vary the discount factor ($\beta$) from 0.95 to 0.80 and the preference parameter for price ($\bar{\alpha}^p$) from −0.2 to −0.3. Estimation assumes each product is a terminal choice and that once a consumer purchases, (s)he exits the market. The DPG assumes consumers track all state variables in forming expectations about the future. There is no differentiation in the transition processes of state variables across products. The IVS method assumes the econometrician estimates the demand model where consumers follow only the inclusive value statistic in forming expectations whereas the full solution method the econometrician assumes that consumer tracks all state variables. $E_p^s$ and $E_\pi^s$ are the average Long term price and profit elasticities associated with a 1% change in price over all $J$ products. $\%\Delta$ correspond to the percent change from teh observed DGP elasticity measures

Mean and standard deviation for 250 simulations

**Table 6** Short term elasticities: Different transition

| | | DGP: One Market: $\beta = 0.95$ | | | | | | DGP: One Market: $\beta = 0.80$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | | DGP | | i: IVS | | iii: Full Solution | | DGP | | i: IVS | | iii: Full Solution | |
| | | $E^s_p$ | $E^s_\pi$ | $\% \Delta E^s_p$ | $\% \Delta E^s_\pi$ | $\% \Delta E^s_p$ | $\% \Delta E^s_\pi$ | $E^s_p$ | $E^s_\pi$ | $\% \Delta E^s_p$ | $\% \Delta E^s_\pi$ | $\% \Delta E^s_p$ | $\% \Delta E^s_\pi$ |
| $\bar{\alpha}^p = -0.2$ | T=50; J=2 | 0.55 | −1.24 | −24.65 | 10.98 | −9.18 | 7.04 | 0.55 | −1.12 | 13.91 | −6.92 | −12.29 | 0.90 |
| | T=50; J=3 | 0.72 | −1.08 | −26.35 | 17.59 | −1.64 | 3.55 | 0.71 | −1.04 | 2.36 | −1.61 | -6.71 | 7.71 |
| | T=50; J=4 | 0.81 | −0.99 | −25.75 | 20.82 | 8.55 | −5.40 | 0.81 | −1.00 | −1.90 | 1.35 | −0.77 | 2.10 |
| | T=50; J=5 | 0.85 | −0.95 | −25.76 | 23.34 | −8.21 | 8.33 | 0.88 | −0.98 | 4.34 | 3.85 | −2.23 | −2.90 |
| $\bar{\alpha}^p = -0.3$ | T=50; J=2 | 0.78 | −0.89 | 4.80 | −4.40 | −8.78 | 9.37 | 0.76 | −0.62 | 14.12 | -17.96 | 7.84 | −18.32 |
| | T=50; J=3 | 1.01 | −0.66 | −4.96 | 7.87 | −5.95 | 11.75 | 0.93 | −0.52 | 2.83 | −5.09 | −9.25 | 21.52 |
| | T=50; J=4 | 1.12 | −0.54 | −9.32 | 19.24 | 5.09 | −9.96 | 1.04 | −0.45 | −2.31 | 4.72 | 0.31 | 2.55 |
| | T=50; J=5 | 1.19 | −0.48 | −13.41 | 33.71 | −3.83 | 8.89 | 1.12 | -0.43 | −6.47 | 16.51 | 2.29 | −3.89 |

This table reports the short term elasticity results of several monte carlo exercises . Exercises vary the discount factor ($\beta$) from 0.95 to 0.80 and the preference parameter for price ($\bar{\alpha}^p$) from −0.2 to −0.3. Estimation assumes each product is a terminal choice and that once a consumer purchases, (s)he exits the market. The DPG assumes consumers track all state variables in forming expectations about the future. There is measurable differentiation in the transition processes of state variables across products. The IVS method assumes the econometrician estimates the demand model where consumers follow only the inclusive value statistic in forming expectations whereas the full solution method the econometrician assumes that consumer tracks all state variables. $E^s_p$ and $E^s_\pi$ are the average Long term price and profit elasticities associated with a 1% change in price over all $J$ products. $\%\Delta$ correspond to the percent change from teh observed DGP elasticity measures

Mean and standard deviation for 250 simulations

2. Obtain $\delta^h_{kt}$ for each product $k$ and period $t$, using the following equation:

$$\delta_{kt} = \frac{F_{k,t}}{(1-\beta)} + \bar{\alpha}^p p_{k,t} \qquad k \in \mathbf{J_t}$$

3. Compute the inclusive value for each consumer:

$$\delta_t = \log\left(\sum_k \exp\left(\delta_{kt}\right)\right)$$

4. Obtain the coefficients through estimation of an AR(1) regression of $\delta_{it}$:
   (a) estimate $\delta_{t+1} = \gamma_0 + \gamma_1 \delta_t + \zeta_t$
   (b) given estimates of $\gamma_1$ and the variance of $\zeta_t$ discretize (N=30) formulate the transition matrix of $\delta_t$ using the Rouwenhorst method

5. Obtain consumer-specific expected value of not purchasing (and hence continuing):

$$EV\left(\delta\right) = \log\left(\exp\left(\delta\right) + \exp\left(\beta\, \mathbf{E}\left[EV(\delta')|\delta\right]\right)\right)$$

   (a) Given the discretized values of $\delta_t$ and the corresponding transition matrix perform a value function iteration to determine $EV\left(\delta\right)$
   (b) Perfrom a linear interpolation of the expected value function back to the estimated values of $\delta_{it}$

6. The model-predicted purchase probability or market share for each product $k$ in each period $t$ is then given as:

$$\hat{s}_{kt} = \frac{\exp\left(\delta_t\right)}{\left[\exp\left(EV\left(\delta_t\right)\right)\right]} \frac{\exp\left(\delta_{kt}\right)}{\exp\left(\delta_t\right)}$$

7. Determine the difference between the observed and simulated market shares at a given parameter set $s_{k,t} - \hat{s}_{kt}\left(\delta_t\right)$

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