Fairness for AUC via Feature Augmentation

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Abstract

We study fairness in the context of classification where the performance is measured by the area under the curve (AUC) of the receiver operating characteristic. AUC is commonly used to measure the performance of prediction models. The same classifier can have significantly varying AUCs for different protected groups and, in real-world applications, it is often desirable to reduce such cross-group differences. We address the problem of how to acquire additional features to most greatly improve AUC for the disadvantaged group. We develop a novel approach, fairAUC, based on feature augmentation (adding features) to mitigate bias between identifiable groups. The approach requires only a few summary statistics to offer provable guarantees on AUC improvement, and allows managers flexibility in determining where in the fairness-accuracy tradeoff they would like to be. We evaluate fairAUC on synthetic and real-world datasets and find that it significantly improves AUC for the disadvantaged group relative to benchmarks maximizing overall AUC and minimizing bias between groups.

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1 Introduction

Algorithms are often the basis of many important decisions in today's business world and society. There are a wide range of applications, including hiring (Liem et al. 2018, De-Arteaga et al. 2019, Lambrecht and Tucker 2019), mortgage lending (Fuster et al. 2020), criminal justice (Berk et al. 2018), and healthcare (Obermeyer et al. 2019), in which algorithms are used to make predictions, which are then used to make decisions, either with or without human supervision. Many such algorithms have been found to be unfair or discriminatory on the basis of legally and socially salient characteristics like race, gender, and age.

Given the importance of achieving fairness across individuals and groups, a wide range of fair algorithms have been proposed. Most of the fairness interventions assume that data is already collected and fixed, and focus on how to design algorithms that are fair. However, if the original data features are collected without recognizing fairness issues, focusing on only the algorithm might not be sufficient. Consider a scenario in which features are selected to maximize accuracy in a population with two groups. Then, it is possible that the features are perfectly predictive for the majority group but entirely uninformative for the minority group. In this case, an algorithmic solution cannot improve the classification accuracy of the minority group. Indeed, a survey of industry practitioners finds that it is at the *data collection* step that practitioners seek guidance (Holstein et al. 2019).

This phenomenon of additional feature acquisition is an everyday business practice where firms obtain more information about their current or prospective customers, e.g. through a credit report. To quote American Express:¹

A major fact of life is that whoever you do business with will likely evaluate your financial status – and do it regularly. Whenever you apply for a loan or sign up for a new cell phone plan, lenders and businesses will make a "hard" pull of your credit report to get a sense of how well or poorly you handle spending and debt payments. Businesses you already have an account with, or others that may want to offer you a "preapproval" deal, can also take a peek at your credit report.

Motivated by this problem, we propose a procedure that uses *feature acquisition or augmentation* (additional feature collection) to improve the predictive performance of disadvantaged groups.² By focusing on the disadvantaged group(s), our approach aims to reduce bias, characterized by the ratio of the area under the receiver operating characteristic curves (AUCs) between protected groups. Our approach, which we call fairAUC, is applicable to a wide variety of classification algorithms and requires only a few data distribution moments of the additional (auxiliary) features. The method is flexible enough to allow decision-makers or managers to determine where in the fairness-accuracy tradeoff they would like to be.

AUC is a non-parametric performance measure that has long been used in binary classification problems, across a wide range of fields, including diagnostic systems, medicine, and in machine learning (Thompson and Zucchini 1989, Bertsimas et al. 2016, Ahsen et al. 2019). AUC is derived from the receiver operating characteristic (ROC) curve, which captures classifier performance in two dimensions by plotting the true positive rate against the false positive rate, by varying the classification threshold. Integrating the area under the ROC curve summarizes the true positive and false positive rates into a single metric, the AUC, which falls between zero and one. AUC is also related to the Mann-Whitney *U*-statistic, and represents the probability that a classifier will rank a randomly chosen positive instance higher than a randomly chosen negative instance.

When should a manager use AUC as a model performance criterion? First, classification algorithms require the manager to set a threshold on scores output by a model to separate the classes. AUC provides a threshold-invariant way to obtain model performance without human judgment regarding appropriate thresholds. AUC integrates across thresholds, and is especially useful in environments where there may be multiple managers, who have different thresholds. Second, AUC is invariant to the base rate, or the proportion of individuals in each class. For data with significant class imbalance, an algorithm would

¹https://www.americanexpress.com/en-us/credit-cards/credit-intel/hard-inquiry-vs-soft-inquiry/

 $^{^{2}}$ A distinct but related literature focuses on feature selection rather than acquisition. With feature selection, the manager already has the data corresponding to all the features and chooses a subset of those to use in an algorithm (Guyon and Elisseeff 2003, Li et al. 2017), whereas with feature acquisition the manager is obtaining auxiliary or additional features from a third-party like a data vendor. The data vendor might reveal either a sample of the records with each of the additional features or provides the manager with summary statistics of the features.

achieve high accuracy by simply always predicting the majority class. However, AUC would not assign this algorithm a high performance measure because the algorithm fails to discriminate between the positive and negative classes. Unlike accuracy, F1, and the area under the precision-recall curve, AUC is robust to changes in the base rate,³ which may vary significantly over time and place (Fawcett 2006). Third, AUC serves as a measure of rank-ordering, which is particularly useful when there are different intensities of intervention available (Kallus and Zhou 2019). For example, a radiologist may set different thresholds for different treatment recommendations based on the outcomes of some tests. Similarly, a bank may set different interest rates depending on credit score and other factors. Thus, the manager would be interested in multiple thresholds, not just one, and AUC can provide an overall characterization across all such thresholds.

Our Contribution

We propose the fairAUC procedure based on feature augmentation to maximally increase the AUC of the disadvantaged group during each round of feature acquisition. It allows the manager to identify new (costly) features to acquire. We provide theoretical guarantees for how fairAUC improves the AUC of each of the groups in a round of feature acquisition. Our theoretical guarantees are directly applicable to the following commonly used models, all of which belong to the class of Generalized Linear Models (GLMs): logistic and multinomial regression, linear SVM, Poisson or negative binomial regression (Nelder and Wedderburn 1972). We evaluate the performance of fairAUC alongside benchmark procedures using synthetic data as well as in real empirical contexts (COMPAS and Diabetes datasets). We find that fairAUC achieves low bias between groups, while obtaining relatively high levels of AUC. Moreover, our approach permits flexibility in determining how many features to acquire, and suggests which ones, based on AUC, fairness, or a weighted combination.

Feature Acquisition using fairAUC

We use a binormal framework to characterize the distribution of a feature and show how Fisher's linear discriminant (FLD) can be used to acquire an additional feature to maximally increase the AUC of the disadvantaged group each feature acquisition round. FLD produces the linear projection which maximizes AUC within this framework (Su and Liu 1993). While other papers have also suggested searching for additional features to increase fairness (Hardt et al. 2016, Chen et al. 2018), we provide specific recommendations on *which features to acquire*. The binormal framework provides us easily interpretable summary statistics and theoretical guarantees.

Figure 1 overviews our proposed fairAUC procedure. fairAUC seeks to improve the AUC of the lower-AUC group, rather than explicitly minimizing bias because the latter does not encourage learning in the long run. Our approach is greedy, focusing on feature acquisition, whereby we start with a set of initial features and then obtain additional features over rounds. The set of features available is summarized by: (a) moments of the data distribution of the auxiliary features, and (b) correlation with the data already collected. The fairAUC procedure chooses the feature that most increases the AUC of the lower-AUC group each round, and proceeds through multiple rounds until a threshold condition is satisfied.

Our research focuses on feature acquisition, which is related to, but distinct from the problem of feature selection. In a feature acquisition problem, the decision maker (manager) has a set of data corresponding to specific individuals. For example, firms have data on customers, and universities have data on applicants. These data correspond to specific features/columns. University applicants' features might include date of birth, gender, citizenship, high school GPA, and test scores. Similarly, a bank offering loans might have features including name, age, income, credit score, number of accounts, and account balance.

Suppose a manager at a bank wants to predict the probability of repayment for a new loan applicant. The manager is also concerned about fairness across groups and finds that these features are very predictive (in the sense of achieving a high AUC) for group a but not for group b. Group, here, could represent race, gender, or any other group of interest. The manager might want to acquire additional features from a data vendor to obtain more fair predictions. To compute the AUC improvement corresponding to each potential feature, the manager provides the data vendor with a set of individual records including an identifier i for each individual,

³When base rates vary, accuracy, F1, and the area under the precision-recall curve will change even if the fundamental characteristics of the classes remain the same (i.e., $\Pr_{\text{train}}[X|Y] = \Pr_{\text{test}}[X|Y]$ but $\Pr_{\text{train}}[Y] \neq \Pr_{\text{test}}[Y]$).



Figure 1: Schematic of Proposed fairAUC Feature Acquisition Procedure

a score S that is the output of a classifier predicting loan repayment, and class Y representing previous loan repayment behavior. The data vendor computes the summary statistics of each available feature, specifically the class-conditional means, variances, and covariances of the features with respect to S. The manager then uses fairAUC to determine which feature to acquire.⁴

Sources of Auxiliary Features

The fairAUC procedure assumes that there are auxiliary features available for acquisition. Some concrete potential sources of additional features include data vendors, like Axciom and Experian, and social media data vendors, like Brandwatch and Grepsr. Third-party data vendors acquire data by buying it, licensing it, or scraping it from public records and can build databases of thousands of features. These features can include personal data, financial data, behavioral data, etc. A number of startups that provide loans (e.g., Earnest, Kreditech) already use such alternative sources of data (e.g., social network data, e-commerce shopping behavior, LinkedIn profiles) to improve their loan provision decisions.⁵ fairAUC enables managers to determine which features to acquire from data vendors based on a small set of summary statistics. Note that fairAUC applies to individual-level data. Table 1 suggests potential auxiliary features and feature sources for several classification problems.

$\mathbf{Problem}^{6}$	Prediction	First-party Data	Auxiliary Features	Source
	Outcome \hat{Y}	Examples $\hat{\mathbf{X}}$	Examples $\hat{\mathbf{Z}}$	
Loan provision	Default	Name, address, SSN,	Work history, college	Data vendor,
		credit history ⁷	major, spending and	social data
			saving behavior,	vendor
			social network data ⁸	
Bail decision	Recidivism	Criminal history,	Spending and saving	Data vendor,
		questionnaire responses ⁹	behavior, credit history,	social data
			social network data	vendor
Hiring	Promotion	Resume, referral,	Social network data	Social data
		interview	engagement	vendor
Extra Medical	Hospital	Biomarker values,	Wearables, social	$Devices^{10}$, social
Attention	Readmission	comorbidities	network data	data vendor

Table 1: Potential Auxiliary Features for Various Classification Problems

More broadly, fairAUC can also be thought about as a framework for determining which features should be collected by the manager or decision maker going forward. For example, consider a government agency that wants to decrease homelessness in the long run across groups. The agency needs to determine whom it should allocate housing to and so the prediction problem is whether someone is going to be homeless two years after receiving housing. Suppose they know a potential set of features (e.g., family support, interests, education) from working with homeless individuals and have some prior distribution over the class-conditional means and variances but have not yet collected data corresponding to those features. Collecting all these features is costly and so the agency seeks to prioritize a single feature to collect. The decision maker can use intuition behind fairAUC to guide which feature to prioritize.

 $^{^{4}}$ A second method to compute the AUC improvement for each feature would be for the manager to acquire all the features available with the data vendor for a (small) subset of the firm's customers. The manager can then compute the class-conditional means, variances and covariances with respect to the score S, and then decide which feature to acquire next using fairAUC. The downside to this strategy is that it could end up being fairly expensive. For example, purchasing all the features from Aspire North, a data vendor, would cost more than \$15,000/1,000 individuals.

⁵https://www.upturn.org/static/files/Knowing_the_Score_Oct_2014_v1_1.pdf

 $^{^{6}}$ Loan provision (Kamiran and Calders 2009, Hardt et al. 2016), bail decision (Larson et al. 2016), hiring (Schumann et al. 2020), extra medical attention (Obermeyer et al. 2019) are problems that have received attention in the fairness literature because of the large impact these decisions have on different groups.

⁷https://apnews.com/article/47693e8cf0ee8328e82cd7ecd06d3df3

⁸https://www.upturn.org/static/files/Knowing_the_Score_Oct_2014_v1_1.pdf

⁹https://www.propublica.org/article/machine-bias-risk-assessments-in-criminal-sentencing

¹⁰e.g., Apple Watch, Fitbit

Advantages of fairAUC

The fairAUC procedure has several appealing aspects. First, it can be used with a variety of classification algorithms. While the firm that is doing the prediction and making the decision (e.g., the bank in the case of offering a loan) has access to its classification algorithm, this algorithm does not need to be shared with the data provider.

In fairAUC, the classification model is retrained during each round by the manager and a score is obtained in the process. This score and class (e.g., prior loan repayment) is then shared with the data provider. The data provider uses the AUC formula to compute the AUC improvement corresponding to each of the available features. The formula only needs the scores, classes, and the new feature vector.

Second, it uses minimal summary statistics of the auxiliary features, rather than requiring full access to the feature matrix. Third, fairAUC does not treat either of the groups as permanently disadvantaged (or advantaged), unlike most research in the fairness literature. Rather, as we proceed with feature augmentation, the *currently higher-AUC group can become the disadvantaged group after the addition of a new feature*. Thus, our goal in each round is to equalize the AUCs by improving the AUC of the *currently disadvantaged* group, preventing reverse discrimination.

Performance of fairAUC

We first characterize how the fairAUC procedure improves the AUC of the lower-AUC group by a minimum threshold amount, thus providing theoretical performance guarantees (under the binormal assumption). Next, we evaluate the performance of fairAUC with synthetic data generated using a systematic data generation procedure proposed by Guyon (2003). We consider three natural benchmarks: minBias, which aims to directly minimize the bias in AUC across groups each round, maxAUC, which ignores fairness to maximize the overall AUC weighted by group size, and a random feature acquisition approach, which may reflect the acquisition of easy-to-acquire data. Each procedure can be used with or without access to the protected class attribute during classification. This is important since certain laws prohibit the use of protected attributes.

Compared to maxAUC, fairAUC achieves significantly greater levels of fairness (in terms of equalizing AUC), with fairly low tradeoffs in AUC. We characterize the accuracy-fairness tradeoff that is achievable using a weighted combination of fairness and AUC objectives, and find that fairAUC obtains low levels of bias without significant sacrifice of overall AUC. Compared to minBias, fairAUC obtains far higher AUCs since minBias fails to incentivize learning.

We also evaluate the four procedures using the COMPAS dataset, a commonly used dataset in fairness studies, auxiliary data purchased from a data vendor, and the Diabetes dataset. We find similar to the synthetic data that fairAUC reduces bias compared to maxAUC with a relatively low tradeoff in AUC. Relative to minBias, fairAUC achieves comparable levels of fairness but with far greater predictive accuracy. Finally, we confirm the robustness of our results using other data generating procedures and classifiers.

2 Related Literature

This paper touches on a few different streams of the fairness in algorithmic systems literature, which addresses questions around bias identification as well as bias reduction.

2.1 Sources of Bias

Researchers have documented a number of causes of bias (Barocas and Selbst 2016) and have documented both human (Mejia and Parker 2021, Benson et al. 2021) and algorithmic discrimination (Fu et al. 2020). It is critically important to understand the source of bias in order to provide guidance to firms and policymakers on how to address bias, since the recommended intervention would depend on the cause. For example, Lambrecht and Tucker (2019) find that advertising on Facebook with the objective of maximizing cost effectiveness inadvertently shows STEM career ads less frequently to women than men, and they report that the source of this bias is that the market bids up the advertising rates to reach women higher than that for men. Thus, in this case market forces are potentially the cause rather than an algorithm. Our study specifically considers that bias can arise due to the nature of data collected, rather than the algorithm. We focus on feature acquisition and its impact on classification performance for members of different protected groups.

2.2 Fairness Criterion

To quantify bias, a measure relevant to the problem must be used. Several fairness criteria have been proposed (Le Quy et al. 2022). In general, the various measures aim to achieve specific criteria, namely independence (Dwork et al. 2012, Kamiran and Calders 2012, Feldman et al. 2015), sufficiency (Chouldechova 2017), and separation (Hardt et al. 2016, Zafar et al. 2017, Kallus and Zhou 2019). It has been shown under mild assumptions that no measure of fairness can simultaneously achieve two of the three criteria (Kleinberg et al. 2017, Chouldechova 2017, Barocas et al. 2019). Therefore, the appropriate fairness criterion depends on the problem of interest (see Appendix A for a comparison of measures). Other measures distinguish between individual versus group fairness, and intertemporal ideas of fairness (Gupta and Kamble 2019). We study group fairness and the criterion we focus on is separation, which recognizes that the protected attribute may be correlated with the target variable. For example, the base rates of loan repayment may differ among groups so a bank may be justified in having different lending rates for different groups (Barocas et al. 2019). The fairness measure we use is related to equalized odds, which achieves separation, in that it is also derived from the ROC curve. Our focus, however, is equalized AUCs, also known as accuracy equity in the literature. As discussed earlier, AUC is not only a metric commonly used to measure classification performance in machine learning but also has many desirable qualities, such as being base rate invariant and threshold invariant.

2.3 Bias Reduction Strategies

Bias reduction strategies can occur prior to (pre-processing), during (in-processing), and after (post-processing) model training. Pre-processing strategies alter the feature space to be uncorrelated with the protected attribute (Kamiran and Calders 2012, Zemel et al. 2013, Feldman et al. 2015, Celis et al. 2020, Shimao et al. 2022). In-processing strategies directly incorporate the fairness constraint into the optimization problem (Dwork et al. 2012, Zafar et al. 2017, Woodworth et al. 2017, Celis et al. 2019). Using AUC as a fairness metric with in-processing has proven challenging and remains an open problem (Celis et al. 2019). Postprocessing strategies occur after classifier training and manipulate the classifier to be uncorrelated with the protected attribute (Hardt et al. 2016). Noriega-Campero et al. (2019) demonstrate that post-processing strategies assume the dataset to be fixed and take an algorithmic approach to reducing bias but practitioners have voiced a need for data collection guidance (Holstein et al. 2019). We take a different approach by developing a procedure for additional feature acquisition, which occurs during the data collection stage. Our solution provides guidance on which additional features should be acquired to improve the AUC of the lower-AUC group and ultimately equalize AUCs across groups.

Fair Feature Acquisition. Our main contribution is to the feature acquisition with fairness considerations literature. This literature addresses *which features* should be acquired and in some cases *which individuals* managers should acquire additional features for. For example, Cai et al. (2020) develop an algorithm to jointly determine which individuals a firm should collect additional features for and then allocate resources to. Noriega-Campero et al. (2019), Bakker et al. (2021), and our work complement this stream of research by determining *which* features to acquire to achieve various measures of fairness. While Noriega-Campero et al. (2019) and Bakker et al. (2021) require the full feature matrix to be known for all individuals in order to determine which feature to acquire, fairAUC requires only a set of summary statistics.

3 Preliminaries and Assumptions

3.1 Preliminaries

We consider a standard binary classification problem with two groups. The dataset consists of N i.i.d. data points $(X_i, A_i, Y_i)_{i=1}^N$ sampled from a distribution \mathcal{D} . Here the input feature $X_i \in \mathbb{R}$, the group $A_i \in \{a, b\}$,

and the class label $Y_i \in \{0, 1\}$. Note that here we specify X_i as a scalar "score" for notational simplicity, whereas our fairAUC procedure accommodates general vectors $X_i \in \mathbb{R}^{d,11}$ We discuss the general case in Section 4.2. Here, and subsequently, we drop the subscript from (X_i, A_i, Y_i) when we do not want to refer to a specific individual. Let $p_{g0}(x) \coloneqq \Pr_{(X,A,Y)\sim\mathcal{D}}[X=x|A=g,Y=0]$ denote the distribution of the input feature belonging to the negative class for each group. Similarly, let $p_{g1}(x) \coloneqq \Pr_{(X,A,Y)\sim\mathcal{D}}[X=x|A=g,Y=1]$ denote the distribution of the input feature belonging to the positive class for each group.

For a data point (X, A, Y), consider a binary classifier r that, for a threshold $\tau \in \mathbb{R}$ is defined as:

$$r(X, A, Y) \coloneqq \begin{cases} 1, & \text{if } X \ge \tau, \\ 0, & \text{if } X < \tau. \end{cases}$$

The true positive rate (TPR) measures the proportion of individuals in the positive class being correctly classified as positive. The false positive rate (FPR) measures the proportion of individuals in the negative class being incorrectly classified as positive. Thus, the TPR and FPR are bounded below by 0 and bounded above by 1. The TPR and FPR of group $g \in \{a, b\}$ can be written as functions of the threshold τ :

$$\operatorname{TPR}_{g}(\tau) \coloneqq \int_{\tau}^{\infty} p_{g1}(x) dx \quad \text{and} \quad \operatorname{FPR}_{g}(\tau) \coloneqq \int_{\tau}^{\infty} p_{g0}(x) dx.$$
 (1)

These two give rise to the ROC curve as follows: the TPR maps the threshold τ to the *y*-axis and the FPR maps τ to the *x*-axis. Formally, for a group *g*, ROC is defined as $\text{ROC}_g(\tau) \coloneqq (\text{FPR}_g(\tau), \text{TPR}_g(\tau))$. The area under the two-dimensional ROC curve (AUC) aggregates the information captured in the TPR and FPR and is defined for a group *g* as follows:

Definition 3.1 (Area under the ROC curve (AUC)).

$$AUC_g \coloneqq \int_0^1 TPR_g(FPR_g^{-1}(x))dx.$$
 (2)

AUC ranges from 0, which occurs when the classifier predicts the opposite of the class label, to 1, which occurs when the classifier can perfectly classify the two classes. An AUC of 0.5 means the classifier cannot distinguish between the two classes and is represented by the diagonal line on the ROC plane.¹² AUC depends on the specific scoring rule r. For now, we consider a general scoring rule and later consider specific scoring rules such as those derived from a linear classifier, a logistic regression classifier, or other generalized linear models.

We measure bias by comparing the AUCs obtained from the groups $g \in \{a, b\}$:

Definition 3.2 (Bias).

$$\operatorname{Bias} \coloneqq 1 - \frac{\min_g(\operatorname{AUC}_g)}{\max_g(\operatorname{AUC}_g)}.$$
(3)

Bias ranges from 0 to 1, with larger values representing greater inequality between groups.

3.2 Class-conditional Means, Variances, and the Binormal Assumption

The class-conditional means and variances of X for each group g are defined as $\mu_{gy} := \mathbb{E}[X|A = g, Y = y]$ and $\sigma_{gy}^2 := \operatorname{Var}[X|A = g, Y = y]$, respectively. The unconditional (class-independent) variance of X for each group is $\operatorname{Var}[X|A = g]$.

To obtain an analytical relationship between moments of the data and AUC, we assume that for each group the input feature follows a binormal distribution (Pesce and Metz 2007), since the ROC (and therefore

¹¹For example, with a logistic regression specified as $\Pr[Y = 1|X] = \frac{\exp(X^{\top}\theta)}{1 + \exp(X^{\top}\theta)}$, the score would be $S = X^{\top}\theta$, and new data that is above a score threshold would be classified as 1.

 $^{^{12}}$ In reality, a classifier is unlikely to obtain an AUC less than 0.5 since making the opposite prediction would improve the AUC. There are cases in which an AUC less than 0.5 can occur. Consider a dataset comprised of two groups where one group makes up the vast majority of the data. Suppose the features of the majority group are negatively correlated with the same features of the minority group. If only a single set of weights is used in classification, the majority group will dominate the determination of the weights and the resulting AUC for the minority group will be worse than random.



Figure 2: (Left) Binormal density plots where the class means of groups a and b are equal but the class-conditional variances of b are greater than those of a. (Right) ROC curves and AUC by group.

its AUC) is known to be robust to departures from this assumption (Hanley 1996). This binormal assumption produces four Gaussian distributions across the two classes and two groups: $\mathcal{N}(\mu_{a0}, \sigma_{a0}^2)$, $\mathcal{N}(\mu_{a1}, \sigma_{a1}^2)$, $\mathcal{N}(\mu_{b0}, \sigma_{b0}^2)$, $\mathcal{N}(\mu_{b1}, \sigma_{b1}^2)$, where 0 and 1 represent the classes, and a and b represent the groups. We assume the means of the positive classes are greater than the means of the negative classes for each group ($\mu_{a1} \ge \mu_{a0}$, $\mu_{b1} \ge \mu_{b0}$) and that the conditional variances are positive. Figure 2 (Left) displays a density plot of two binormal distributions for which the class-conditional variances within each group are equal but the class-conditional variances for group a are smaller than those for group b.

Incorporating the binormal assumption, TPR and FPR from Equation (1) for group g can be written as:

$$\operatorname{TPR}_{g}(\tau) = 1 - \Phi\left(\frac{\tau - \mu_{g1}}{\sigma_{g1}}\right) \quad \text{and} \quad \operatorname{FPR}_{g}(\tau) = 1 - \Phi\left(\frac{\tau - \mu_{g0}}{\sigma_{g0}}\right).$$
(4)

Where $\Phi(\cdot)$ represents the standard normal cumulative distribution function. We can now express the AUC defined in Equation (2) as a function of the class-conditional means and variances of each group:

$$AUC_g = \Phi\left(\frac{\mu_{g1} - \mu_{g0}}{\sqrt{\sigma_{g0}^2 + \sigma_{g1}^2}}\right).$$
(5)

Figure 2 (Right) displays the ROC curves and their associated AUCs from the binormal distributions shown in Figure 2 (Left). The diagonal line represents random guessing.

4 Methodology

Many papers consider the unconditional distributions of the input feature for each group (Corbett-Davies and Goel 2018, Chen et al. 2018, Emelianov et al. 2022). It may be expected that higher variance, flatter unconditional distributions generate higher AUCs since greater spread provides more information. Suppose X takes just one value (is deterministic). Then the unconditional variance is zero and the classifier learns nothing from this data. Given this example, one may believe that a higher unconditional variance corresponds to better classification. However, this thought experiment conflates the separation of means with variance. Analyzing the unconditional distribution mixes together base rates, class-conditional means, and class-conditional variances, obscuring the relationship between the data and AUC. Section 7.1 formalizes the previous ideas.

Here we highlight which features of the data distributions pin down AUC. Equation (5) informs us that increasing the difference in class means and decreasing the class-conditional variances increase the AUC. Figure 2 (Left) and (Right) visualize an example in which the differences in class means are equal between the two groups but the conditional variances of group b are larger than those of group a. Because of the larger conditional variances, the AUC of b is lower than the AUC of a. Finally, the base rate π_g does not appear in the AUC formula, reinforcing the idea that differences in base rates between the two groups will not contribute to bias with respect to AUC.

4.1 Strategies for Additional Feature Acquisition

We consider strategies for selecting additional costly features for classification with fairness considerations. One natural strategy would be to select features that minimize the bias across groups. We consider a greedy strategy, minBias, that chooses the feature minimizing the difference between AUCs across groups in each round. However, such a strategy fails to incentivize additional learning; when features that are relatively uninformative for both groups minimize bias, it will select those features rather than features that improve AUC.

Another strategy is to select features that improve the AUC of the disadvantaged group (group with lower AUC). We develop a procedure which does exactly this, noting that with a greedy feature acquisition procedure over multiple rounds, the group that is (dis)advantaged may change across rounds. Note that such a procedure does not guarantee a reduction in bias because it is possible that the additional feature is even more predictive for the higher-AUC group or dramatically increases the AUC of the disadvantaged group much more than for the advantaged group.

Thus, the natural question is: given the data we have, what additional features(s) should we acquire to *most improve the AUC of the currently disadvantaged group*? Because AUC is calculated using a scalar score, we must select a dimensionality reduction method which aggregates our existing data with the additional feature(s) into one dimension. Here we use Fisher's linear discriminant (FLD) as our dimensionality reduction method because it generates the linear projection which maximizes AUC (Su and Liu 1993). Later, we will broaden the set of classification models to extend beyond FLD to the class of Generalized Linear Models (GLMs).

4.1.1 Fisher's Linear Discriminant (FLD)

We make a few simplifying assumptions. First, we assume only *one* additional feature can be acquired in each round and develop a greedy strategy. We relax this assumption in Section 8.5. Second, we assume the existing input feature the manager has can be represented in one dimension (i.e., the output of a score function). Third, we assume that a binormal distribution provides a reasonable approximation for the features. Using FLD, we determine the benefit of a new feature to the AUC of the disadvantaged group under consideration (denoted g^*).

Let *n* denote the number of individuals in the disadvantaged group g^* . Consider a dataset $(X_i, A_i, Y_i)_{i=1}^n$ where we have already collected one (non-protected) feature $X_i \in \mathbb{R}$ for each individual *i*. Since our focus is only on a single group, we drop the group subscript notation used in the previous sections. Moreover, assume we have access to an auxiliary feature $(Z_i)_{i=1}^n$, which we could choose to acquire.

We seek to determine the benefit to g^* of acquiring $(Z_i)_{i=1}^n$. For each outcome class $y \in \{0,1\}$, the class-conditional mean vector and covariance matrix of $(X_i, Z_i)_{i=1}^n$ are:

$$\boldsymbol{\mu}_{\boldsymbol{y}} \coloneqq \begin{bmatrix} \mu_{X,y} \\ \mu_{Z,y} \end{bmatrix} = \begin{bmatrix} \mathbb{E}[X|Y=y] \\ \mathbb{E}[Z|Y=y] \end{bmatrix} \quad \text{and} \quad \boldsymbol{\Sigma}_{\boldsymbol{y}} \coloneqq \begin{bmatrix} \sigma_{X,y}^2 & \rho_y \sigma_{X,y} \sigma_{Z,y} \\ \rho_y \sigma_{X,y} \sigma_{Z,y} & \sigma_{Z,y}^2 \end{bmatrix}.$$

Where $\sigma_{X,y}^2 = \operatorname{Var}[X|Y = y]$, $\sigma_{Z,y}^2 = \operatorname{Var}[Z|Y = y]$, and $\rho_y = \frac{\operatorname{Cov}[X,Z|Y=y]}{\sigma_{X,y}\sigma_{Z,y}}$. Note that ρ_y represents the class-conditional correlation of X and Z and not the unconditional correlation.

Let \boldsymbol{w} represent a potential projection direction that projects (X_i, Z_i) for each individual $i \in [n]$ to \mathbb{R} , combining the two features into a single value. Then the projected class mean $\tilde{\mu}_y \in \mathbb{R}$ is defined as $\tilde{\mu}_y \coloneqq \boldsymbol{w}^\top \boldsymbol{\mu}_y$ and the projected class-conditional variance $\tilde{\sigma}_y^2 \in \mathbb{R}$ is defined as $\tilde{\sigma}_y^2 \coloneqq \boldsymbol{w}^\top \boldsymbol{\Sigma}_y \boldsymbol{w}$. The FLD objective function is known to maximize AUC (Su and Liu 1993). In terms of the projected means and variances, the FLD objective is: $J(\boldsymbol{w}) \coloneqq \frac{(\tilde{\mu}_1 - \tilde{\mu}_0)^2}{\tilde{\sigma}_0^2 + \tilde{\sigma}_1^2}$. In terms of the pre-projection mean vectors and covariance matrices, it is:

$$J(\boldsymbol{w}) = \frac{\boldsymbol{w}^\top (\boldsymbol{\mu_1} - \boldsymbol{\mu_0}) (\boldsymbol{\mu_1} - \boldsymbol{\mu_0})^\top \boldsymbol{w}}{\boldsymbol{w}^\top (\boldsymbol{\Sigma_0} + \boldsymbol{\Sigma_1}) \boldsymbol{w}}$$

The projection direction \boldsymbol{w} which maximizes $J(\boldsymbol{w})$ can be found by solving a generalized eigenvalue problem (Duda et al. 2012). The optimal linear projection direction (when $\Sigma_0 + \Sigma_1$ is invertible) is given by:

$$\boldsymbol{w}^{\star} = (\boldsymbol{\Sigma}_{0} + \boldsymbol{\Sigma}_{1})^{-1} (\boldsymbol{\mu}_{1} - \boldsymbol{\mu}_{0}).$$
(6)

Plugging $\tilde{\mu}_y = \boldsymbol{w}^{\star \top} \boldsymbol{\mu}_y$ and $\tilde{\sigma}_y^2 = \boldsymbol{w}^{\star \top} \boldsymbol{\Sigma}_y \boldsymbol{w}^{\star}$ into Equation (5) yields the AUC of the optimal linear combination of input features (X, Z):

AUC(X, Z) =
$$\Phi\left(\sqrt{(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^\top (\boldsymbol{\Sigma}_0 + \boldsymbol{\Sigma}_1)^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)}\right).$$
 (7)

The benefit to the disadvantaged group of acquiring Z is the difference between this new value of AUC and the previous value of AUC that used only X.

More generally, one can consider other scoring rules such as those derived from Generalized Linear Models (GLMs). A GLM is specified by a vector of weights $\theta \in \mathbb{R}^2$ and an increasing and invertible link function $\psi \colon \mathbb{R} \to \mathbb{R}$. It generates scores $\psi^{-1}(\theta_1 X + \theta_2 Z)$, and given a threshold τ , it labels a sample X as 1 if and only if

$$\psi^{-1}(\theta_1 X + \theta_2 Z) > \tau.$$

For instance, if ψ is the logit function, then the GLM is a logistic regression classifier which predicts 1 if and only if

$$\frac{\exp\left(\theta_1 X + \theta_2 Z\right)}{1 + \exp\left(\theta_1 X + \theta_2 Z\right)} > \tau.$$

It turns out that the ROC curve of the GLM G specified by θ and ψ is the same as that for the linear classifier L which uses the linear projection $\theta_1 X + \theta_2 Z$ and, hence, the AUC of G and L is the same.¹³ Thus, Equation (7) is also the highest AUC achievable by a GLM that uses X and Z to predict Y.

So far, Z has represented an arbitrary feature available for acquisition. When given a choice over many possible features, which feature Z maximizes Equation (7) for a given X?

4.2 fairAUC Procedure

We now present our fairAUC procedure, for which Equation (7) serves as the backbone. It relies on knowing only a few summary statistics of the data. It can be used with data vendors who provide costly features. Alternatively, managers may collect a small sample of additional features and estimate the benefit of each using this strategy prior to collecting the features for all individuals.

It begins by taking in the data the manager has for N individuals $(\mathbf{X}_i, A_i, Y_i)_{i=1}^N$, the manager's scoring algorithm r, and the level of acceptable bias ε . As before, we drop the subscript from (\mathbf{X}_i, A_i, Y_i) when we do not refer to a specific individual. The input data $\mathbf{X} \in \mathbb{R}^d$, the group $A \in \{a, b\}$, and the class $Y \in \{0, 1\}$.

The manager aggregates the features in X into a single score $S \in \mathbb{R}$ using a fixed scoring algorithm r. Our framework allows for any scoring algorithm r. When the data is binormal, FLD generates the optimal AUC. Otherwise, FLD is meant to serve as a heuristic and one can replace it with a different scoring algorithm r. We show in Section 8.4 that different classifiers, including nonlinear ones, end up acquiring nearly all the same features, although at times in different orders. Furthermore, r may or may not use the protected attribute A depending on the context. For instance, FICO is prohibited from using characteristics like race, gender, and marital status in producing its credit score. We refer to using A as using separate classifiers for each group and not using A as using only a single classifier for both groups.

In each round t of the fairAUC procedure, there is a bias identification step, an algorithmic step, and a feature acquisition step. In the bias identification step, we first calculate the AUC for each of the groups from the scores S. The bias of the model is calculated from the AUCs of the two groups. If the bias is smaller than a given tolerance level ε , the manager does not need to take any intervention to reduce bias. However, if the bias is larger than ε , the manager acquires one additional feature per round. The group with lower AUC is referred to as the currently disadvantaged group, which can vary over the rounds of feature acquisition, and is denoted g^* .

In the algorithmic step, fairAUC aims to acquire the feature that most increases the AUC of g^* using the FLD heuristic explained in the previous section (i.e., Equation (7)). As previously discussed, FLD generates a linear combination of features which maximizes AUC and requires only summary statistics to calculate. Let $(\mathbf{Z}_i)_{i=1}^N$ where $\mathbf{Z}_i \in \mathbb{R}^{d'}$ represent the auxiliary features available for acquisition. Let m = d + d' capture the total number of features that exist in the data the manager owns and in the auxiliary data available for

¹³This is because G at threshold τ makes the same prediction as L at threshold $\psi(\tau)$ – see Claim 7.8.

acquisition. Let $\{1, \ldots, d'\}$ denote the indices of all auxiliary features that are available for acquisition. In any given round t, we use $Q(t) \subset \{1, \ldots, d'\}$ to denote the set of auxiliary features acquired so far. Initially, we have that $Q(0) = \emptyset$. Hence, the set of features available is $[d'] \setminus Q$. We assume that the cost of each feature is the same and that a feature is acquired for all N individuals.

For group g^* , the manager obtains the class-conditional means of each of the features available for acquisition in $[d'] \setminus Q$ as well as the class-conditional covariance matrices of each of the available features and the score S, requiring the manager to share S with the data vendor each round and Y the first round.¹⁴ We relax this requirement in Section 8.7, where we assume the score S to be uncorrelated with the auxiliary features, eliminating the need to share S. The conditional means and covariances inform how valuable each of the additional features is to the manager in terms of increasing the AUC of g^* . The manager acquires the feature which maximizes the AUC of group g^* .

In the feature acquisition step, Q(t) is updated to include this new feature. The feature acquired is concatenated with the existing dataset and becomes the input to the next round. Procedure 1 formalizes each iteration of fairAUC. To simplify the notation, let \hat{X} denote the collection of $(X_i)_{i=1}^N$, and similarly \hat{Z} the collection of $(Z_i)_{i=1}^N$. Z_i denotes a row vector of features for each individual, Z^j denotes a column feature vector across all individuals, and \hat{Z} denotes the collection of features for all individuals.

Procedure 1: fairAUC (*t*-th iteration)

Input: data owned $(\mathbf{X}_i, A_i, Y_i)_{i=1}^N$, scoring algorithm r, bias threshold ε , set of acquired features Q(t), data available for acquisition $(\mathbf{Z}_i)_{i=1}^N$; **Output:** Q(t+1);if A cannot be used then $\boldsymbol{S} \coloneqq r(\hat{\boldsymbol{X}}, \boldsymbol{Y});$ else $| \boldsymbol{S} \coloneqq r(\hat{\boldsymbol{X}}, \boldsymbol{A}, \boldsymbol{Y});$ for group $g \in \{a, b\}$ do $g^{\star} \coloneqq \arg\min_{g} \operatorname{AUC}_{g}(S)$ (Disadvantaged group); $\text{Bias} \coloneqq 1 - \frac{\min_g(\text{AUC}_g(S))}{\max_g(\text{AUC}_g(S))} \text{ (Definition 3.2) };$ if $Bias > \varepsilon$ then for feature $\mathbf{Z}^j \in \hat{\mathbf{Z}}, j \notin Q(t)$ do for feature Z^{j} , group g^{\star} , and score S, obtain class-conditional means, μ_{0}, μ_{1} , and covariance matrices, Σ_0, Σ_1 (Summary Statistics by Group Subroutine); $h(\boldsymbol{S},\boldsymbol{Z}^{j}) \coloneqq \Phi\left(\sqrt{(\boldsymbol{\mu}_{1}-\boldsymbol{\mu}_{0})^{\top}(\boldsymbol{\Sigma}_{0}+\boldsymbol{\Sigma}_{1})^{-1}(\boldsymbol{\mu}_{1}-\boldsymbol{\mu}_{0})}\right);$ $j^{\star} := \arg\max_{j} h(\boldsymbol{S}, \boldsymbol{Z}^{j});$ acquire feature $Z^{j^{\star}}$: return $Q(t+1) := Q(t) \cup \{j^\star\};$ else no intervention;

4.3 Theoretical Guarantees on Improvement of AUC by fairAUC

In this section, we present our theoretical result on the fairAUC procedure in the binormal framework.

Different classifiers that use the same set of features can have different AUCs. We define the AUC of group g at the start of iteration t, as the highest AUC that a GLM can achieve for samples in group g using features Q(t). For instance, if the link function ψ is the logit function (corresponding to logistic regression), then AUC of group g is equal to the highest AUC achieved by a logistic regression classifier using Q(t). The specific choice of ψ does not affect the definition because it turns out that changing the link function ψ does not change the ROC curve of a GLM and, hence, does not change its AUC (see Claim 7.8).

At a high level, our result uses the fact that if S is an FLD-based score on the acquired features Q(t), then the AUC achieved by scores S can be computed using Equation (5) and it is equal to the AUC of group

 $^{^{14}}$ Note that in the case of working with a data vendor the fairAUC procedure assumes that the data vendor also knows A. In practice, this information may need to be shared. A benefit of the fairAUC procedure is that it does not require the manager to share X with the data vendor, only S. It also does not require the data vendor to share more than a few summary statistics.

Subroutine: Summary Statistics by Group

Input: feature available for acquisition Z, group g, existing score S; Output: class-conditional mean vectors μ_0, μ_1 , class-conditional covariance matrices Σ_0, Σ_1 ; for class $y \in \{0, 1\}$ do $n := n_{A=g,Y=y}$; $\mu_y := \begin{bmatrix} \bar{S}_y \\ \bar{Z}_y \end{bmatrix} = \begin{bmatrix} \frac{1}{n} \sum_{i: \stackrel{A_i=g}{Y_i=y}} S_i \\ \frac{1}{n} \sum_{i: \stackrel{A_i=g}{Y_i=y}} Z_i \end{bmatrix}$; $\Sigma_y := \begin{bmatrix} \sigma_{S,y}^2 & \rho_y \sigma_{S,y} \sigma_{Z,y} \\ \rho_y \sigma_{S,y} \sigma_{Z,y} & \sigma_{Z,y}^2 \end{bmatrix}$ where $\sigma_{S,y}^2 = \frac{1}{n-1} \sum_{i: \stackrel{A_i=g}{Y_i=y}} (S_i - \bar{S}_y)^2, \sigma_{Z,y}^2 = \frac{1}{n-1} \sum_{i: \stackrel{A_i=g}{Y_i=y}} (Z_i - \bar{Z}_y)^2$, and $\rho_y = \frac{1}{(n-1)\sigma_{s,y}\sigma_{Z,y}} \sum_{i: \stackrel{A_i=g}{Y_i=y}} (S_i - \bar{S}_y)(Z_i - \bar{Z}_y)$; return $\mu_0, \mu_1, \Sigma_0, \Sigma_1$

g (as defined above). To prove our result, we provide fairness guarantees for fairAUC that uses FLD-based scores S. We begin by analyzing the improvement in AUC of the disadvantaged group g^* . In each iteration $t \in \mathbb{N}$ of fairAUC, where the AUC for g^* is bounded away from 1, and there is at least one auxiliary feature present which has "low" class-conditional covariances with the current scores S for g^* and has "bounded" class-conditional variances and means for g^* , fairAUC is guaranteed to improve the AUC of g^* by at least a constant in iteration t.

Proposition 4.1. (Theoretical guarantee on fairAUC; informal statement. See Theorem 7.5 for formal version). In the binormal framework, if the summary statistics subroutine outputs the unbiased means and covariances of the queried features and S is the FLD-based score, then we can provably guarantee that in each iteration of the fairAUC procedure, the AUC value of the disadvantaged group g^* increases by at least

$$\max_{\ell} \frac{1}{4} \cdot \gamma^{3/2} \cdot \beta_{\ell}^2 \cdot (1 - \delta_{\ell})^2.$$

Here ℓ runs over unacquired features, γ is the distance of the current AUC value of g^* from 1, β_{ℓ} is the difference in the normalized class-conditional means of the ℓ th unacquired feature on g^* (Equation (17)), and δ_{ℓ} is the absolute value of the normalized class-conditional covariance between the FLD-based score S and the ℓ th unacquired feature on g^* (Equation (18)).

The theorem implies that the improvement is more when γ and β_{ℓ} are large and δ_{ℓ} is close to 0 for at least one unacquired feature, and less when γ is close to 0 or for all features either β_{ℓ} is close to 0 or δ_{ℓ} is close to 1. To see why the improvement value may depend on these quantities, note that:

- 1. If γ is close to 0, then the AUC of the disadvantaged group is close to 1, which is its maximum value, and hence, cannot increase significantly. In contrast, when we have data with features that are not as informative, then the potential improvement γ and actual improvement are higher.
- 2. Intuitively, the difference between class-conditional means (β_{ℓ}) helps create separation between the two classes, and including features with high separation improves AUC. If β_{ℓ} is close to 0, then Equation (5) tells us that the classifier using the ℓ th unacquired feature to predict the outcome has a low AUC, and so the ℓ th unacquired feature is not a good predictor and does not increase the AUC significantly.
- 3. The normalized covariance provides a measure of dependence between the new feature to be acquired and the score that summarizes existing features. This is related to mutual information, and since FLD is a linear discriminant, covariance provides a characterization of the mutual information. When δ_{ℓ} is close to 1, then the ℓ th unacquired feature is highly correlated with the score derived from the existing features, and so does not add additional information.

There are a few points to note. First, while a specific feature ℓ might create separation for the disadvantaged group, it might also create such separation for the advantaged group. Recall that fairAUC by design only

focuses on improving the performance of the disadvantaged group. However, we will prove in Theorem 7.6 in the Appendix similar bounds on the increase in AUC for the advantaged group.

Second, note that it is possible for the AUC of the currently disadvantaged group to exceed that of the advantaged group due to this feature acquisition. If that happens, the definition of disadvantaged group changes for the next round.

Third, the lower bound in Proposition 1 applies when fairAUC uses FLD as the scoring rule. However, once the features are selected, the data manager can retrain a GLM of their choice, say logistic regression. If the lower bound in Proposition 4.1 is $\Delta(t)$ at the *t*-th iteration, then the AUC of the logistic regression classifier trained by the data manager at the *t*-th iteration is at least additively $\Delta(t)$ larger than the AUC of the logistic regression classifier from the previous iteration.¹⁵

4.4 fairAUC with Bias Guarantees: noisy fairAUC

The fairAUC procedure can increase bias if the acquired feature is even more informative for the advantaged group or if the AUC of the disadvantaged group greatly overshoots the AUC of the advantaged group after the acquisition of a new feature. More formally, suppose we have data \mathbf{X} and we acquire feature \mathbf{Z} so that our new data is $\mathbf{X}' := (\mathbf{X}, \mathbf{Z})$. We can then have $\operatorname{Bias}(\mathbf{X}') > \operatorname{Bias}(\mathbf{X})$, where $\operatorname{Bias}(\mathbf{X})$ and $\operatorname{Bias}(\mathbf{X}')$ are the values of bias obtained using features \mathbf{X} and \mathbf{X}' respectively. If our aim is to always decrease bias, one strategy is to alter fairAUC by adding a noisy version of feature \mathbf{Z} instead. The noisy version of \mathbf{Z} introduces noise to the group that would have higher AUC after adding \mathbf{Z} . Specifically, for this group, we use

$$\mathbf{Z}' \coloneqq \lambda \mathbf{Z} + (1 - \lambda) N(0, 1)$$

where $0 \le \lambda \le 1$ indicates how much weight to place on **Z**.

The intuition for noisy fairAUC comes from the following idea: When a new feature is added, if we add pure noise to the feature (say a random normal variable of zero mean and some variance), then the AUC of the classifier with the data including the new feature is monotonically decreasing in the variance of the noise added. The intuition is more readily apparent if we consider an extreme case: consider adding pure noise as the new feature to the advantaged group a (so $\lambda_a = 0$), and therefore completely ignoring the newly acquired feature. In that case, the AUC of the advantaged group would remain the same. In contrast, for the disadvantaged group b, for any $0 < \lambda_b < 1$, the AUC would increase by the addition of the new feature. Given this disparate impact on the AUCs across the groups, bias would decrease during this round of feature acquisition.

More generally, we can apply this intuition to the less extreme case of adding noise to the new feature only for the advantaged group. By choosing an appropriate weight to place on the noise, we obtain a provable guarantee that in noisy fairAUC, the bias between the two groups always (weakly) decreases during each round of feature acquisition (see Section 7.3). Moreover, the noisy fairAUC algorithm retains the desirable property of the original fairAUC algorithm that the AUC of both groups is also guaranteed to (weakly) increase in each round as features are acquired. Thus, this approach of adding noise addresses the concern that bias could potentially increase in fairAUC. Note that this noisy fairAUC algorithm does not restrict the features in any way, or impose any additional assumptions beyond what is required for fairAUC.

What is the tradeoff then? Of course, the bias guarantee in noisy fairAUC will come at the cost of a reduction in the AUC improvement for the advantaged group (but it is important to note that it will never result in a *decrease* in AUC for any group). By adding noise to the advantaged group's acquired feature, we are reducing the AUC of the advantaged group relative to the fairAUC case, where no noise was added. We detail how the main theoretical result that all groups see an increase in AUC during a feature acquisition round continues to hold with noisy fairAUC (see Remark 7.14 in Section 7.3). The above tradeoff positions noisy fairAUC at a different point on the fairness-accuracy curve relative to fairAUC. This positioning allows a manager to decide which algorithm to use based on how much they value the requirement that bias be non-increasing.

¹⁵The same also holds for GLMs with link functions ψ other than logit. This is because the link function does not affect the AUC of a GLM classifier – see Claim 7.8.

5 Empirical Results

We compare four feature acquisition strategies, namely fairAUC, maxAUC, minBias, and random. During each round of feature acquisition, the fairAUC procedure selects the feature that most improves AUC for the group with lower AUC according to FLD. The maxAUC procedure selects the feature that most improves the *overall weighted AUC* using FLD weighted by group size (see Appendix B for the maxAUC procedure). The minBias procedure selects the feature that minimizes the bias between the two groups. The random procedure selects a feature at random, and represents a baseline in which the manager collects additional data in an uninformed manner.

5.1 Synthetic Data

We use the data generation strategy proposed by Guyon (2003) for the controlled benchmarking of variable selection algorithms in binary classification problems. We generate N = 20,000 individuals with 50 non-protected continuous binormally distributed features, one binary protected feature (group), and a binary outcome (class). Of the 50 features, half are *informative* in that the class-conditional distributions of each of the features have means that are separated from each other. The remaining features are *uninformative* random noise features. Group a constitutes 70% and group b 30%. The base rate of positive class labels in both groups is 25%.

For the synthetic data, there is nothing fundamentally different between the two groups besides the number of individuals in each group so any difference in predictive performance stems from the feature acquisition procedure. We set the level of acceptable bias $\varepsilon = 10^{-6}$ to demonstrate the various procedures over many rounds. We acquire 10 additional features and use logistic regression for classification. We randomly select one feature to represent the data the manager begins with (Round 0) for classification.

5.1.1 Synthetic Data Results

Figure 3 (Top) compares the fairAUC, maxAUC, minBias, and random procedures in terms of group-wise AUCs when the protected attribute is used in the scoring function (i.e., each group has a separate classifier). Under fairAUC, the initially disadvantaged minority group b quickly obtains predictive performance equal to majority group a. Under maxAUC, group b's AUC always trails group a's AUC even though separate classifiers are trained for each group. The minBias procedure quickly reduces bias and maintains low bias but fails to select informative features. fairAUC and maxAUC outperform the random and minBias procedures in terms of group-wise AUC. Figure 8 in Section 8.2 compares the procedures when the protected attribute is not used. The overall patterns among the procedures remain the same.

We graph the accuracy-fairness tradeoff (where accuracy is measured by AUC) in Figure 3 (Center) that results from using fairAUC rather than maxAUC. Ideally, a procedure generates points in the lower right of the graph, i.e., low bias and high AUC. The dotted lines connect the corresponding rounds between fairAUC and maxAUC. All of the lines have a positive slope, indicating that fairAUC reduces bias but at the cost of overall AUC. For fairAUC, we observe that the bias does not monotonically decrease but rather jumps around. After all the rounds are complete, the manager can evaluate the accuracy (AUC) versus bias tradeoff, according to their requirements. If the manager requires a lower bias, they could choose the round that corresponds to the lowest level of bias (Round 7), whereas if they prefer to tradeoff a higher level of bias for a higher AUC, they might choose Round 10. The crucial aspect is that the feature acquisition procedure provides the manager with a flexible set of options at various points on the accuracy-bias spectrum. The minBias procedure, as expected, produces low bias values but at the cost of significantly lower AUC. The random procedure generates bias values between fairAUC and maxAUC but at far worse AUC values than either. Figure 9 in Section 8.2 graphs the tradeoff when the protected attribute A is not used and we observe the same patterns.

Finally, we evaluate convex combinations of the fairness and maximum AUC objectives to generate a Pareto frontier for bias and overall AUC. Figure 3 (Bottom) shows the intermediate bias and overall AUC values that can be achieved by altering the weight of the two objectives over different feature augmentation rounds when A is used.¹⁶ Full weight on the fairness objective represents fairAUC and full weight on the

 $^{^{16}}$ For earlier rounds, many weight combinations select the same feature acquisition strategy, resulting in overlap.

AUC objective represents maxAUC. The manager therefore also has flexibility in determining how much weight to give to each of the objective functions. Figure 10 in Section 8.2 graphs the Pareto frontier when A is not used and shows a similar pattern. maxAUC generates far higher levels of bias when A is not used in classification.

The results highlight a number of pitfalls that can occur in data collection and prediction algorithm design. First, collecting data to maximize overall AUC or accuracy can inadvertently hurt the minority group. This can occur even when the two groups are equally separable and separate classifiers are trained for each group. Second, a strategy that aims to only minimize bias can result in the collection of features that are not predictive for either group.

5.1.2 Robustness of fairAUC to Assumptions

fairAUC uses FLD as a heuristic to determine which feature to acquire. Recall that FLD is the linear classifier that maximizes AUC when the data is binormally distributed. We test the robustness of fairAUC to other data generating processes, specifically using a different distribution which can accommodate a range of possible probabilities in Section 8.3. We find that the results are robust to using data that follow various gamma distributions. Next, we have specified the FLD framework and the associated theoretical results as applying to the class of Generalized Linear Models (GLMs). Therefore, we test the robustness of the results to using nonlinear classifiers like random forest and nonlinear SVM, in place of a linear classifier, logistic regression (Section 8.4). Here too, we find similar results that our proposed fairAUC procedure achieves much lower bias across groups, while obtaining overall AUC at a slightly lower to similar level as maxAUC. Finally, fairAUC requires back and forth data exchange between the manager and the data vendor to calculate covariances. One strategy to simplify the procedure even further is to assume independence between the auxiliary features and the firm's data. Table 9 in Section 8.7 shows that out of the ten features acquired when correlations are accounted for, eight are acquired when the correlations are assumed to be zero. Ignoring the class-conditional correlations generally results in the acquisition of the same features but in a less efficient order. We next apply the fairAUC method to two real-world datasets from two distinct application areas. namely criminal justice and health care.

5.2 Application: Predicting Violent Recidivism

5.2.1 COMPAS Dataset

The dataset covers 6,172 criminal defendants from Broward County, Florida and contains information on their COMPAS score, demographics (gender, race, age), criminal history, and whether they actually recidivated within a two-year period after release. Our target variable of interest is violent recidivism and the protected attribute is age^{17} (under 25 vs. 25+). Those under 25 represent 33% of the data and have a 14% violent recidivism rate while those over 25 have a 10% violent recidivism rate.

5.2.2 Data Pre-processing

We take log of the numerical variables in the dataset (e.g., number of priors) to reduce the impact of outliers. We also convert the categorical variables (e.g., race) into binary variables. We do not use the risk assessment levels or decile scores generated by COMPAS as inputs since they are the outcome variables, i.e. essentially what we seek to predict.

Suppose a judge has data on gender (initial independent variable), age group (protected attribute), and whether each defendant violently reoffended within two years of being released (outcome variable). Defendants under 25 years of age are the initially disadvantaged (lower-AUC) group based on the data the judge has. Given that features are costly to acquire, our focus is on which additional feature a judge should collect to better predict the likelihood of violent recidivism for defendants under 25.

 $^{^{17}}$ Note that we define age as age at charge, which is different from the age recorded in the ProPublica dataset. ProPublica records defendants' age in 2016, the year the data was collected, rather than the age at charge. We calculate age at charge by subtracting date of birth from the date the defendant went to jail.



Figure 3: (Top) AUC by group over feature acquisition rounds using different feature acquisition strategies and using the protected attribute. The vertical error bars represent 95% confidence intervals on the AUCs. (Center) Comparison of accuracy-fairness tradeoff among feature acquisition strategies using the protected attribute. (Bottom) Pareto frontier of convex combinations of the fairness and AUC objectives for several rounds of feature acquisition using the protected attribute.





5.2.3 COMPAS Results

We use logistic regression as the classifier and compare the fairAUC, maxAUC, minBias, and random procedures over ten rounds. Figure 4 plots the performance of the four procedures. The AUCs have fairly large error bars but the means follow the pattern seen in the synthetic data. fairAUC improves the AUC for the group of defendants under 25 and decreases bias while maxAUC does not close the gap between the two groups.

5.2.4 Acquiring Additional Features from a Data Vendor

In the previous section, we treat the features in the COMPAS dataset as features that can be acquired. In this section, we treat the COMPAS data as given and purchase additional features from a data vendor. Because the COMPAS dataset includes names and dates of birth for the defendants, it is possible to purchase additional features for these individuals. We note that the data available for purchase is current (2022) and not dated back to when the defendant was arrested (2013/2014), making it inappropriate for the actual prediction task at hand. The goal here is to demonstrate the practical feasibility of obtaining data from another source and combining it with first-party data.

We acquire data from Aspire North, a data vendor that works with small- to medium-sized businesses. The vendor sells a core set of roughly 550 features for 80/1,000 individuals and charges more for specific features, such as ethnicity, net worth, and spending in different categories. Purchasing all available features exceeds 15,000/1,000 individuals. See Table 8 in Section 8.6 for additional details about the features.

We provide the data vendor the following information: name, birthday, zip codes in Broward county for the first pass of matching, and zip codes in Florida for the second pass of matching. Of the 6,172 defendants in the COMPAS dataset, the data vendor successfully matched 1,679 individuals. Some individuals may have changed their name, moved out of state, or it could be that the data vendor does not have data on all individuals.¹⁸ Defendants under 25 represent 30% of the data and have a 10.3% violent recidivism rate while those over 25 have a 9.6% violent recidivism rate. In addition to the core set of features, we purchase nine additional-fee features.

5.2.5 Data Vendor Feature Acquisition Results

To compare the performance of the four procedures, we begin with the COMPAS features as the initial data and acquire features from the purchased dataset. Where there are missing values in the acquired data, we replace the values with the means from each group. We use logistic regression as the classifier. Figure 5

 $^{^{18}\}mathrm{When}$ given name and address, Experian claims a match rate of 85%.

plots the performance of the four algorithms. For this dataset, the initial data is more predictive of violent recidivism for the defendants under 25 years of age. Compared to maxAUC, fairAUC greatly reduces bias between the two groups.



Figure 5: Predicting Violent Recidivism using Protected Attribute (Age) with Data Vendor Features

5.3 Application: Predicting Hospital Readmission

Finally, we test the various procedures on a second completely distinct dataset in a healthcare application using the publicly available Diabetes dataset (Strack et al. 2014).¹⁹ The dependent variable examined is hospital readmission within 30 days, and the explanatory (predictor) variables include reason for admission, time in hospital, intervention with drugs, number of medications, and other health-related data, as well as demographic data on gender, race, and age. After cleaning the data, there are 45,715 observations. We take log of the numeric variables to reduce the impact of outliers and convert the categorical variables to indicator variables.

We find that when examining race as a protected group (77% Caucasian, 23% Non-Caucasian), our fairAUC procedure reduces bias, while obtaining a high AUC for both Caucasian and Non-Caucasian groups, compared to the maxAUC procedure, which obtains much higher AUC for the group of Caucasian individuals. Our real-world applications span across multiple domains, lending external validity to the broad range of applications where fairAUC can be useful.

6 Conclusion

We propose fairAUC, an approach to feature acquisition that helps achieve fairness in the AUC measure. Our approach, which can incorporate a wide variety of classification algorithms, aims to improve the performance of the lower-AUC group. First, using a theoretical analysis we show provable AUC improvements for the disadvantaged group. Second, we test our approach using synthetic data as well as in real-world contexts and find that our approach performs well in reducing bias, while also increasing AUC for both the disadvantaged and advantaged groups.

While our method has many advantages, it is not without limitations. First, our method applies to cases with binary outcome labels, although in principle it could be extended to a multiclass classification problem. Second, if two ROC curves cross, then one classifier performs better in one region of ROC space and the other classifier performs better in the other region of ROC space. Our approach would only consider the overall AUC. In practical situations, we might want to weight false positives and true positives differently or consider

¹⁹https://archive.ics.uci.edu/ml/datasets/diabetes+130-us+hospitals+for+years+1999-2008





a notion of partial AUC. In practice, the algorithm can be altered to account for such asymmetric weights. Third, our procedure assumes the underlying data distributions are approximately binormal. While fairAUC is meant to provide guidance as a heuristic, large deviations from normality may undermine its effectiveness. However, we do find that fairAUC continues to perform well with different gamma distributions and in real-world data.

We trust that this paper is a first step in identifying and directly addressing fairness as it relates to the data collection process and AUC. This work complements other work that focuses on data collection through rows (Cai et al. 2022), rather than features. We expect that more broadly these areas and their interaction will be further investigated in future research.

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7 Proofs

7.1 Proof That AUC Can Be Non-monotone in Unconditional Variance

In this section, we formalize the idea that the unconditional variance does not inform AUC by writing the unconditional variance as a function of the conditional variances, where $\pi_g \coloneqq \Pr[Y = 1|A = g]$ represents the proportion of observations from the positive class for group g. The conditional variances, σ_{g1}^2 and σ_{g0}^2 , and the unconditional variance, $\operatorname{Var}[X|A = g]$, can be written as:

$$\sigma_{g_1}^2 = \operatorname{Var}[X|A = g, Y = 1] \\ = \mathbb{E}[X^2|A = g, Y = 1] - \mathbb{E}[X|A = g, Y = 1]^2 \\ = \int x^2 p_{g_1}(x) dx - \mu_{g_1}^2,$$
(8)

$$\sigma_{g0}^{2} = \operatorname{Var}[X|A = g, Y = 0]$$

= $\mathbb{E}[X^{2}|A = g, Y = 0] - \mathbb{E}[X|A = g, Y = 0]^{2}$
= $\int x^{2} p_{g0}(x) dx - \mu_{g0}^{2},$ (9)

and

$$\operatorname{Var}[X|A = g] = \mathbb{E}[X^2|A = g] - \mathbb{E}[X|A = g]^2$$

$$= \pi_g \int x^2 p_{g1}(x) dx + (1 - \pi_g) \int x^2 p_{g0}(x) dx - (\pi_g \mu_{g1} + (1 - \pi_g) \mu_{g0})^2.$$
(10)

It follows from Equations (8), (9), and (10) that:

$$\operatorname{Var}[X|A=g] = \pi_g (1-\pi_g)(\mu_{g1}-\mu_{g0})^2 + \pi_g \sigma_{g1}^2 + (1-\pi_g)\sigma_{g0}^2.$$
(11)

When we hold the difference in class means and base rate constant, different combinations of σ_{g0}^2 and σ_{g1}^2 can produce the same unconditional variance in Equation (11). According to the binormal AUC formula (Equation (5)), these combinations of σ_{g0}^2 and σ_{g1}^2 do not all map to the same AUC for group g. Indeed, the same unconditional variance can be mapped to multiple AUCs. The left half of Table 2 (Constant $\operatorname{Var}[X|A = g]$) shows a numerical example of a single unconditional variance mapping to multiple AUCs for different conditional variances. The right half of Table 2 (Increasing $\operatorname{Var}[X|A = g]$) shows that there is not a monotonic relationship between the unconditional variance and AUC.

Table 2: Unconditional Variance and AUC

Cor	nstant	$\operatorname{Var}[X A]$	l = g]	Incr	Increasing $\operatorname{Var}[X A=g]$				
σ_{g0}^2	σ_{g1}^2	Var	AUC	σ_{g0}^2	σ_{g1}^2	Var	AUC		
10	1	18.80	0.82	2	4	19.60	0.95		
4	2.5	18.80	0.94	12	3	20.80	0.75		
2	3	18.80	0.98	4	8	23.20	0.80		
Note	<i>Note:</i> $\pi_g = 0.8$ and $\mu_{g1} - \mu_{g0} = 10$.								

Observation 7.1 (Non-informativeness of Unconditional Variance). The ranking of the unconditional variance between groups is not informative of the ranking of AUC between groups. For groups a and b, if Var[X|A = a] > Var[X|A = b], AUC_a can be greater than, equal to, or less than AUC_b .

Proof of Observation 1. Set $\pi_a = \pi_b = \pi$, $\mu_{a1} = \mu_{b1} = \mu_1$, and $\mu_{a0} = \mu_{b0} = \mu_0$. Then, using Equation (11), $\operatorname{Var}[X|A=a] > \operatorname{Var}[X|A=b]$ implies:

$$\pi \sigma_{a1}^2 + (1 - \pi) \sigma_{a0}^2 > \pi \sigma_{b1}^2 + (1 - \pi) \sigma_{b0}^2.$$
(12)

Further, suppose that $\pi < 0.5$ and $\mu_1 \neq \mu_0$. Consider the following two cases that demonstrate that AUC_a can be greater than or less than AUC_b.

1. Let $\sigma_{a0}^2 = \sigma_{a1}^2 = \sigma_a^2$ and $\sigma_{b0}^2 = \sigma_{b1}^2 = \sigma_b^2$. It follows from Equation (12) that $\sigma_a^2 > \sigma_b^2$ so AUC_a = $\Phi\left(\frac{\mu_1-\mu_0}{\sqrt{2\sigma_a^2}}\right) < \Phi\left(\frac{\mu_1-\mu_0}{\sqrt{2\sigma_b^2}}\right) = AUC_b$. Here, we also use the fact that $\mu_1 \neq \mu_0$ and that $\Phi(\cdot)$ is a monotonically increasing function.

2. Let $\sigma_{a0}^2 = \sigma_{a1}^2 = \sigma_a^2$. It follows from Equation (12) that:

$$\sigma_a^2 > \pi \sigma_{b1}^2 + (1 - \pi) \sigma_{b0}^2.$$

Let

$$\sigma_a^2 = \pi \sigma_{b1}^2 + (1 - \pi) \sigma_{b0}^2 + \varepsilon$$
(13)

where $\varepsilon > 0$. We want to find conditions under which AUC_a \geq AUC_b. It follows from Equation (5) that AUC_a \geq AUC_b when $2\sigma_a^2 \leq \sigma_{b1}^2 + \sigma_{b0}^2$ (since $\Phi(\cdot)$ is a monotonically increasing function). Incorporating Equation (13), the AUC condition requires:

$$\sigma_{b1}^2 + \sigma_{b0}^2 \ge 2\pi\sigma_{b1}^2 + 2(1-\pi)\sigma_{b0}^2 + 2\varepsilon,$$

which simplifies to:

$$\sigma_{b1}^2 \ge \sigma_{b0}^2 + \frac{2\varepsilon}{1 - 2\pi} \tag{14}$$

when $\pi < 0.5$.

Note that the smaller class needs to have higher variance for Equation (14) to hold. Class-conditional variances are weighted in the expected overall unconditional variance but not weighted in the AUC formula. The closer we are to class balance (i.e., $\pi = 0.5$) the greater the difference in class-conditional variances we need for AUC_a \geq AUC_b.

7.2 Proof of Proposition 4.1

In this section, we prove Proposition 4.1, which lower bounds the AUC-improvement for the disadvantaged group in each iteration of fairAUC. We also prove an analogous result for the advantage group (Theorem 7.6).

Toward this, we analyze the fairAUC procedure in the binormal framework for features (Su and Liu 1993) (where the features follow a normal distribution conditioned on the class and the protected group). We show that if fairAUC uses FLD-based scores S (Equation (15)), then in each iteration $t \in \mathbb{N}$, where the AUC for the disadvantaged group is bounded away from 1 and there is at least one auxiliary feature which has "low" class-conditional covariances with the current scores S and has "bounded" class-conditional variances and means, fairAUC improves the AUC of the disadvantaged group by at least a constant in iteration t (Theorem 7.5).

From Section 4.2, recall that there are a total of m features, out of which, the decision-maker initially has access to d acquired features:

$$X \coloneqq (X^1, X^2, \dots, X^d) \in \mathbb{R}^d, \qquad (\text{Acquired features})$$

and has the option to augment d' := m - d auxiliary features:

$$Z \coloneqq (Z^1, Z^2, \dots, Z^{d'}) \in \mathbb{R}^{d'}.$$
 (Auxiliary features)

The *m* features $X \cup Z$, together with class label *Y* and group *A*, are assumed to follow the following binormal framework in this section.

Definition 7.2 (Binormal framework). The *m* features $X \cup Z$, the class label Y, and the group label A are distributed according to a distribution \mathcal{D} over $\mathbb{R}^d \times \mathbb{R}^{d'} \times \{0,1\} \times \{a,b\}$, such that for each $y \in \{0,1\}$ and $g \in \{a,b\}$, conditioned on A = g and Y = y, the *m* features, follow a *m*-variate normal distribution with an invertible covariance matrix. (Note that conditioned on A = g and Y = y different features can be correlated with each other.)

From the distribution \mathcal{D} (in Definition 7.2), $N \in \mathbb{N}$ independent samples are drawn to construct a dataset D before starting the fairAUC procedure. The summary statistics subroutines (Subroutines SSR and SSR2) use D, every time they are queried, to compute the approximations to first and second moments of the distribution \mathcal{D} ; we assume that these approximations have a negligible error (Assumption 1).

Assumption 1. Assume that the sample means and covariances computed by two summary statistics subroutines (Subroutines SSR and SSR2) are equal to the corresponding true means and covariances of draws from \mathcal{D} .

Since the subroutines use independent samples from \mathcal{D} , where the features follow a normal distribution, from the concentration inequalities of the normal distribution (Tropp 2015), we expect the samples means and covariances of the features on D to be "good approximations" of the true means and covariances of the features on \mathcal{D} for large N.

At each iteration, fairAUC acquires one auxiliary feature. For each $t \in [d']$, let $Q(t) \subseteq [d']$ denote the set of all auxiliary features acquired before the start of the *t*-th iteration; where we have $Q(1) \coloneqq \emptyset$. Further, let X(t) denote the tuple of all features in Q(t) and the *d* features (X^1, X^2, \ldots, X^d) , i.e.,

$$X(t) \coloneqq (X^1, X^2, \dots, X^d) \ \cup \ \left(Z^\ell\right)_{\ell \in Q(t)}.$$

Note that because $Q(1) = \emptyset$, X(1) = X.

We need to define the AUC of X(t) for a group $g \in \{a, b\}$ (Definition 7.4) before stating our results. Note that X(t) is always of the form $X \cup \{Z\}_{\ell \in Q(t)}$, i.e., $k \ge 0$ auxiliary features augmented to X. We restrict our definition of the AUC (Definition 7.4) to such sets of features.

Procedure: fairAUC (*t*-th iteration)

Input: Data owned $(X_i, A_i, Y_i)_{i=1}^N$, [d'], indices acquired $Q(t) \subseteq [d']$, and data acquired $(Z_i^{\ell})_{i \in [N], \ell \in Q(t)}$ **Output:** Set $Q(t+1) \subseteq [d']$ of the auxiliary features augmented

for group $g \in \{a, b\}$ do Query $\Sigma_{0}^{(g)}, \Sigma_{1}^{(g)}, \mu_{0}^{(g)}, \mu_{1}^{(g)} = SSR(X \cup (Z^{\ell})_{\ell \in Q(t)}, A, g)$ Compute $\Delta \mu^{(g)} := (|\mu_{11}^{(g)} - \mu_{01}^{(g)}|, \dots, |\mu_{1d}^{(g)} - \mu_{0d}^{(g)}|)$ Compute $\Sigma^{(g)} := \Sigma_{0}^{(g)} + \Sigma_{1}^{(g)}$ Initialize $S := (0)_{i=1}^{N}$ for $i \in [N]$ do if $A_{i} = a$ then | Set $S_{i} := (\Delta \mu^{(a)})^{\top} (\Sigma^{(a)})^{-1} X_{i}$ // Compute FLD scores else | Set $S_{i} := (\Delta \mu^{(b)})^{\top} (\Sigma^{(b)})^{-1} X_{i}$ // Compute FLD scores

for group $g \in \{a, b\}$ do

$$\begin{bmatrix} \text{Compute AUC}_{g}(X) \coloneqq \Phi\left(\sqrt{(\mu_{1}^{(g)} - \mu_{0}^{(g)})^{\top}(\Sigma_{0}^{(g)} + \Sigma_{1}^{(g)})^{-1}(\mu_{1}^{(g)} - \mu_{0}^{(g)})}\right) \\ g(t) \coloneqq \arg\min_{g \in \{a,b\}}(\text{AUC}_{g}(X))$$

// Find disadvantaged group

for auxiliary feature $\ell \in [d']$ do

// For group g(t) query: class-conditional means $\mu_0, \mu_1 \in \mathbb{R}^2$, and // covariance matrices $\Sigma_0, \Sigma_1 \in \mathbb{R}^{2 \times 2}$ between score S and auxiliary feature Z^{ℓ} . Query $\Sigma_0, \Sigma_1, \mu_0, \mu_1 = SSR2(\ell, g(t), S)$

Compute
$$\operatorname{AUC}_{g(t)}(S, Z^{\ell}) \coloneqq \Phi\left(\sqrt{(\mu_1 - \mu_0)^{\top}(\Sigma_0 + \Sigma_1)^{-1}(\mu_1 - \mu_0)}\right)$$

 $i \coloneqq \arg\max_{\ell \in [d']} \operatorname{AUC}_{q(t)}(S, Z^{\ell}) \ Q(t+1) = Q(t) \cup \{i\}$

return Q(t+1).

Definition 7.3 (AUC of generalized linear models on group g). Given $k \ge 0$ auxiliary features, say Z^1, Z^2, \ldots, Z^k , acquired features $X \in \mathbb{R}^d$, a vector $w \in \mathbb{R}^{d+k}$ and an increasing and invertible link function

Subroutine: SSR (summary statistic subroutine)

Input: Acquired features $\{X_i^j\}_{i \in [N], j \in [d+t]}$, protected attributes $\{A_i\}_{i=1}^N$, group $g \in \{a, b\}$ **Output:** class-conditional mean vectors $\mu_0, \mu_1 \in \mathbb{R}^d$, class-conditional covariance matrices $\Sigma_0, \Sigma_1 \in \mathbb{R}^{d \times d}$

for class $y \in \{0,1\}$ do

Compute $n \coloneqq \sum_{i} \mathbb{I}[A_{i} = g, Y_{i} = y]$ // Total elements with $A_{i} = g$ and $Y_{i} = y$ Compute $\mu_{y} \coloneqq \frac{1}{n} \left[\sum_{i: A_{i} = g, Y_{i} = y} X_{i}^{1}, \dots, \sum_{i: A_{i} = g, Y_{i} = y} X_{i}^{d} \right]$ // Empirical mean of X when $(A_{i}, Y_{i}) = (g, y)$ Compute matrix $\Sigma_{y} \in \mathbb{R}^{d \times d}$, where for all $\ell, k \in [d]$ // Empirical covariance of X when $(A_{i}, Y_{i}) = (g, y)$ $(\Sigma_{y})_{\ell,k} \coloneqq \frac{1}{n-1} \sum_{i: A_{i} = g, Y_{i} = y} (X_{i}^{\ell} - (\mu_{y})_{\ell})(X_{i}^{k} - (\mu_{y})_{k})$

return $\mu_0, \mu_1, \Sigma_0, \Sigma_1$

Subroutine: SSR2 (Summary statistic subroutine - 2)

Input: auxiliary feature index $\ell \in [d']$, group g, score $\{S_i\}_{i=1}^N$ (Also, has access to all auxiliary features $\{Z_i\}_{i=1}^N$)

Output: Class-conditional mean vectors $\mu_0, \mu_1 \in \mathbb{R}^2$, class-conditional covariance matrices $\Sigma_0, \Sigma_1 \in \mathbb{R}^{2 \times 2}$

 $\begin{array}{l} \text{for } class \; y \in \{0,1\} \; \text{do} \\ \text{Compute } n \coloneqq \sum_{i} \mathbb{I}[A_{i} = g, Y_{i} = y] \\ \text{Compute } \mu_{S,y} \coloneqq \frac{1}{n} \sum_{i:\; A_{i} = g, Y_{i} = y} S_{i} \\ \text{Compute } \mu_{Z,y} \coloneqq \frac{1}{n} \sum_{i:\; A_{i} = g, Y_{i} = y} Z_{i}^{\ell} \\ \text{Compute } \Sigma_{y} \coloneqq \begin{bmatrix} \sigma_{S,y}^{2} & \rho_{y} \\ \rho_{y} & \sigma_{Z,y}^{2} \end{bmatrix} \text{ where} \\ \\ \sigma_{S,y}^{2} \coloneqq \frac{1}{n-1} \sum_{i:\; A_{i} = g, Y_{i} = y} (S_{i} - \mu_{S,y})^{2}, \\ \sigma_{Z,y}^{2} \coloneqq \frac{1}{n-1} \sum_{i:\; A_{i} = g, Y_{i} = y} (Z_{i}^{\ell} - \mu_{Z,y})^{2}, \text{ and} \\ \\ \rho_{y} \coloneqq \frac{1}{n-1} \sum_{i:\; A_{i} = g, Y_{i} = y} (S_{i} - \mu_{S,y}) \cdot (Z_{i}^{\ell} - \mu_{Z,y}) \end{array}$

return $\mu_0, \mu_1, \Sigma_0, \Sigma_1$

 $\psi \colon \mathbb{R} \to \mathbb{R}$, consider a classifier C that given threshold $\tau \in \mathbb{R}$, predicts

$$\mathbb{I}\left[\psi^{-1}\left(\sum_{i=1}^d w_i X^i + \sum_{i=1}^k w_{d+i} Z^i\right) > \tau\right].$$

Then the AUC of C for group $g \in \{a, b\}$, denoted by

$$\operatorname{AUC}_g(w,\psi,X,Z^1,\ldots,Z^k),$$

is the area under the ROC curve of C when samples ((X, Z), Y, A) are drawn from \mathcal{D} conditioned on A = g.

Definition 7.4 (AUC for group g). Given $k \ge 0$ auxiliary features, say Z^1, Z^2, \ldots, Z^k , acquired features $X \in \mathbb{R}^d$, and a group $g \in \{a, b\}$, define the AUC of (X, Z^1, \ldots, Z^k) for group g as

$$\operatorname{AUC}_g(X, Z^1, \dots, Z^k) \cong \max_{w, \psi} \operatorname{AUC}_g(w, \psi, X, Z^1, \dots, Z^k),$$

where the maximum is over all $w \in \mathbb{R}^{d+k}$ and all increasing and invertible functions $\psi \colon \mathbb{R} \to \mathbb{R}$.

Using Definition 7.4, we can formalize the "FLD-based" score that fairAUC uses in this section. For all samples in group $g \in \{a, b\}$ (i.e., $i \in [N]$, with A = g), define the scores $S(t) \in \mathbb{R}$ used in fairAUC as the projection of X(t) that maximizes the AUC of the best generalized linear model on group g (see Definition 7.3):

$$S(t) \coloneqq (\psi^{\star})^{-1} (\langle w^{\star}, X(t) \rangle), \text{ where } w^{\star}, \psi^{\star} \coloneqq \operatorname{argmax}_{w,\psi} \operatorname{AUC}_{g(t)}(w, \psi, X(t)).$$
(15)

One can show that S(t) is equivalent to the projection obtained using FLD on each group; this uses the fact that the data follows the binormal framework (Definition 7.2; see Su and Liu (1993)).

Now, we restrict out attention to a particular iteration $t \in \mathbb{N}$. We define certain quantities that show up in our results (Theorem 7.5). Suppose $g(t) \in \{a, b\}$ is the disadvantaged group in the *t*-th iteration. For each auxiliary feature $\ell \in [d'] \setminus Q(t)$, let $\Delta v_{\ell}^{(t)}$ be the absolute difference of its class conditional means (on g(t)), i.e.,

$$\Delta v_{\ell}^{(t)} \coloneqq \left| \mathbb{E}[Z^{\ell} \mid Y = 1, A = g(t)] - \mathbb{E}[Z^{\ell} \mid Y = 0, A = g(t)] \right|.$$
(16)

Next, using $\Delta v_{\ell}^{(t)}$ define the following quantities for each auxiliary feature $\ell \in [d'] \setminus Q(t)$

$$\beta_{\ell}^{(t)} \coloneqq \frac{\Delta v_{\ell}^{(t)}}{\sqrt{\sum_{y \in \{0,1\}} \operatorname{Var}[Z^{\ell} \mid Y = y, A = g(t)]}},\tag{17}$$

$$\delta_{\ell}^{(t)} \coloneqq \frac{1}{\Delta v_{\ell}^{(t)}} \cdot \left| \sum_{y \in \{0,1\}} \operatorname{Cov}[S(t), Z^{\ell} \mid Y = y, A = g(t)] \right|.$$
(18)

Finally, define $\gamma^{(t)}$ as

$$\gamma^{(t)} \coloneqq 1 - \operatorname{AUC}_{g(t)}(X(t)).$$

We prove Theorem 7.5.

Theorem 7.5 (Effect of fairAUC on the AUC of the disadvantaged group). Suppose that the *m* features $X \cup Z$, class label Y, and protected group A follow the binormal framework (Definition 7.2). Further, assume that two summary statistics subroutines satisfy Assumption 1. Then, for all iterations $t \in [d']$ and all auxiliary features $\ell \in [d'] \setminus Q(t)$, it holds that

$$\operatorname{AUC}_{g(t)}(S(t), Z^{\ell}) - \operatorname{AUC}_{g(t)}(X(t)) > \frac{1}{4} \cdot \left(\gamma^{(t)}\right)^{3/2} \cdot \left(\beta_{\ell}^{(t)} \cdot (1 - \delta_{\ell})^{(t)}\right)^{2}.$$
(AUC increment on selecting Z^{ℓ})

Further, the auxiliary feature $i \in [d'] \setminus Q(t)$ selected by fairAUC in the t-th iteration satisfies

$$\operatorname{AUC}_{g(t)}(X(t), Z^{i}) - \operatorname{AUC}_{g(t)}(X(t)) \geq \max_{\ell \in [d'] \setminus Q(t)} \frac{1}{4} \cdot \left(\gamma^{(t)}\right)^{3/2} \cdot \left(\beta_{\ell}^{(t)} \cdot (1 - \delta_{\ell})^{(t)}\right)^{2}.$$
(AUC increment by fairAUC) (19)

Some remarks are in order:

- 1. (Dependence on $\gamma^{(t)}$). As $AUC_{g(t)}(X(t))$ approaches 1 (i.e., $\gamma^{(t)}$ approaches 0), the lower bound in Equation (19) approaches 0. This is expected because when $AUC_{g(t)}(X(t))$ is close to 1, which is its maximum value, each auxiliary feature can only increment the AUC for g(t) by a small amount.
- 2. (Dependence on $\beta_{\ell}^{(t)}$). If $\Delta |v_{\ell}^{(t)}|$ is small or $\sum_{y \in \{0,1\}} \operatorname{Var}[Z^{\ell} \mid Y = y, A = g(t)]$ is large, then Equation (5) tells us the classifier which uses Z^{ℓ} to predict the class Y has a low AUC, i.e., Z^{ℓ} is not a "good predictor" of Y. This is captured by $\beta_{\ell}^{(t)}$ in Equation (19). To see this, observe that when $\Delta |v_{\ell}^{(t)}|$ is small or $\sum_{y \in \{0,1\}} \operatorname{Var}[Z^{\ell} \mid Y = y, A = g(t)]$ is large, $\beta_{\ell}^{(t)}$ is small. Thus, the increment in the AUC is also small.
- 3. (Dependence on $\delta_{\ell}^{(t)}$). To gain some intuition about the dependence on $\delta_{\ell}^{(t)}$, consider the extreme case, where Z^{ℓ} is identical to S(t). This maximizes the class-conditional covariances of Z^{ℓ} and S(t) (on A = g(t)) subject to a fixed value of variance of Z^{ℓ} . Thus, it also maximizes $\delta_{\ell}^{(t)}$. However, in this case, any linear combination of X(t) and Z^{ℓ} is identical to some linear combination of X(t) (and vice-versa).²⁰ Thus, $AUC_{g(t)}(X(t), Z^{\ell}) = AUC_{g(t)}(X(t))$. Intuitively, Z^{ℓ} does not provide any new information.

Theorem 7.5 shows the effect of fairAUC on the AUC of the disadvantaged group. Our next result (Theorem 7.6), captures the effect of fairAUC on the AUC of the advantaged group. Suppose $\hat{g}(t) \in \{a, b\}$ is the advantaged group at the *t*-th iteration. Theorem 7.6 provides a lower bound in the improvement on the AUC of the advantaged group in the *t*-th iteration in terms of quantities $\Delta \hat{v}_{\ell}^{(t)}$, $\hat{\beta}_{\ell}^{(t)}$, $\hat{\delta}_{\ell}^{(t)}$, and $\hat{\gamma}^{(t)}$ (Equations (20) to (23)); these are equivalent to $\Delta v_{\ell}^{(t)}$, $\beta_{\ell}^{(t)}$, $\delta_{\ell}^{(t)}$, $\alpha q \gamma^{(t)}$ in Theorem 7.5, except the disadvantaged group g(t) in the definitions changes to the advantaged group $\hat{g}(t)$.

²⁰This uses the fact that S(t) is a linear combination of X(t). Since Z^{ℓ} is identical to S(t), Z^{ℓ} is also linear combination of X(t).

Formally, we define $\Delta \hat{v}_{\ell}^{(t)}$, $\hat{\beta}_{\ell}^{(t)}$, $\hat{\delta}_{\ell}^{(t)}$, and $\hat{\gamma}^{(t)}$ as follows.

$$\Delta \hat{v}_{\ell}^{(t)} \coloneqq \left| \mathbb{E}[Z^{\ell} \mid Y = 1, A = \hat{g}(t)] - \mathbb{E}[Z^{\ell} \mid Y = 0, A = \hat{g}(t)] \right|, \tag{20}$$

$$\hat{\beta}_{\ell}^{(t)} \coloneqq \frac{\Delta v_{\ell}}{\sqrt{\sum_{y \in \{0,1\}} \operatorname{Var}[Z^{\ell} \mid Y = y, A = \hat{g}(t)]}},\tag{21}$$

$$\hat{\delta}_{\ell}^{(t)} \coloneqq \frac{1}{\Delta v_{\ell}^{(t)}} \cdot \left| \sum_{y \in \{0,1\}} \operatorname{Cov}[S(t), Z^{\ell} \mid Y = y, A = \hat{g}(t)] \right|,$$
(22)

$$\hat{\gamma}^{(t)} \coloneqq 1 - \operatorname{AUC}_{\hat{g}(t)}(X(t)).$$
(23)

Theorem 7.6 (Effect of fairAUC on the AUC of the advantaged group). Suppose that the *m* features $X \cup Z$, class label Y, and protected group A follow the binormal framework (Definition 7.2). Further, assume that two summary statistics subroutines satisfy Assumption 1. Then, for all iterations $t \in [d']$ the auxiliary feature $i \in [d'] \setminus Q(t)$ selected by fairAUC in the t-th iteration satisfies

$$\operatorname{AUC}_{\hat{g}(t)}(X(t), Z^{i}) - \operatorname{AUC}_{\hat{g}(t)}(X(t)) \geq \frac{1}{4} \cdot \left(\hat{\gamma}^{(t)}\right)^{3/2} \cdot \left(\hat{\beta}_{\ell}^{(t)} \cdot (1 - \hat{\delta}_{\ell})^{(t)}\right)^{2}.$$
(AUC increment by fairAUC) (24)

At a first glance, the lower bound in Theorem 7.6 may appear to be equivalent to Theorem 7.5. The difference is that fairAUC is guaranteed to pick the best feature for the disadvantaged group, but it may not pick the best feature for the minority group. Thus, while the lower bound in Theorem 7.5 is large if any auxiliary feature $\ell \in [d'] \setminus Q(t)$ has large $\beta_{\ell}^{(t)}$ and small $\delta_{\ell}^{(t)}$, Theorem 7.6 requires $\beta_i^{(t)}$ to be large and $\delta_i^{(t)}$ to be small for the particular feature $i \in [d'] \setminus Q(t)$ selected by fairAUC.

7.2.1 Preliminaries

In this section, we present three lemmas which will be used in the proof of Theorem 7.5.

Lemma 7.7 (Expression for optimal AUC (Su and Liu 1993, Corollary 3.1)). Consider two random variables $X \in \mathbb{R}^d$ and $Y \in \{0,1\}$, which are distributed according to a joint distribution \mathcal{D} , such that for all $y \in \{0,1\}$, conditioned on Y = y, X follows a multivariate normal distribution with mean $\mu_y \in \mathbb{R}^d$ and covariance matrix $\Sigma_y \in \mathbb{R}^{d \times d}$:

for all
$$y \in \{0, 1\}$$
, $X \mid Y = y \sim \mathcal{N}(\mu_y, \Sigma_y)$.

Let $\Delta \mu \coloneqq |\mu_1 - \mu_0|$ and $\Sigma \coloneqq \Sigma_0 + \Sigma_1$. Then, the maximum AUC of a generalized linear model that takes X as input and predicts Y is $\Phi\left(\sqrt{\Delta\mu\Sigma^{-1}\Delta\mu}\right)$.

Proof. Proof of Lemma 7.7 Corollary 3.1 in Su and Liu (1993) proves that the maximum AUC of any linear classifier that takes X as input and predicts Y is $\Phi\left(\sqrt{\Delta\mu\Sigma^{-1}\Delta\mu}\right)$.

This also extends to generalized linear models because the AUC of a generalized linear model is independent of its link function. In particular, given a generalized linear model, we can choose its link function as the identity function without changing its AUC. This converts the generalized linear model to a linear classifier, for which the result follows by Corollary 3.1 in Su and Liu (1993). Let $G_{w,\psi}$ be the generalized linear model with weights $w \in \mathbb{R}^d$ and increasing and invertible link function $\psi : \mathbb{R} \to \mathbb{R}$.

Claim 7.8 (AUC of a generalized linear model is independent of its link function). For any $w \in \mathbb{R}^d$, there is a value $v \in [0,1]$ such that, for any increasing and invertible link function $\psi \colon \mathbb{R} \to \mathbb{R}$, the AUC of $G_{w,\psi}$ is v.

Proof. Proof of Claim 7.8 Let $I: \mathbb{R} \to \mathbb{R}$ be the identity function (i.e., for all $x \in \mathbb{R}$, $I(x) \coloneqq x$). (For simplicity, we assume that the domain and range of ψ are \mathbb{R} . The same proof holds for general case by limiting the threshold t to the domain or range of ψ as appropriate.) For any value $t \in \mathbb{R}$, $G_{w,\psi}$ with

threshold t makes the same predictions as $G_{w,I}$ with threshold $\psi(t)$: This holds because for any increasing and invertible function ψ

$$w^{\top}X > \psi(\tau)$$
 if and only if $\psi^{-1}(w^{\top}X) > \tau$. (25)

(The "if" follows by applying ψ to both sides of $\psi^{-1}(w^{\top}X) > \tau$ and using the fact that ψ is increasing. "Only if" follows by applying ψ^{-1} to both sides of $w^{\top}X > \psi(\tau)$ and using that ψ^{-1} is increasing.)

Let $\text{TPR}_{\psi}(t)$ and $\text{TPR}_{\psi}(t)$ be the true positive rate and the false positive rate of $G_{w,\psi}$ at threshold t. Similarly, let $\text{TPR}_I(t)$ and $\text{TPR}_I(t)$ be the true positive rate and the false positive rate of $G_{w,I}$ at threshold t. From the above observation (Equation (25)) it follows that for all thresholds $t \in \mathbb{R}$

$$\operatorname{TPR}_{\psi}(t) = \operatorname{TPR}_{I}(\psi(t)) \text{ and } \operatorname{FPR}_{\psi}^{-1}(t) = \psi^{-1}\left(\operatorname{FPR}_{I}^{-1}(\psi(t))\right)$$

Now the claim follows by using the definition of AUC in Equation (2):

AUC of
$$G_{w,\psi} = \int_0^1 \text{TPR}_{\psi} \left(\text{FPR}_{\psi}^{-1}(z) \right) dz$$

$$= \int_0^1 \text{TPR}_I \left(\psi \left(\psi^{-1} \left(\text{FPR}_I^{-1}(z) \right) \right) \right) dz$$

$$= \int_0^1 \text{TPR}_I \left(\text{FPR}_I^{-1}(z) \right) dz$$

$$= \text{AUC of } G_{w,I}.$$

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This completes the proof of Claim 7.8 and also of Lemma 7.7.

Lemma 7.9 ("Inverting" $\Phi(\sqrt{\cdot})$). For all $\alpha \ge 0$ and $\gamma > 0$, if $\Phi(\sqrt{\alpha}) < 1 - \gamma$ then it holds that

$$\alpha < -2 \cdot \ln\left(2\gamma\right).$$

Proof. Proof of Lemma 7.9 We use the fact that for all $x \in \mathbb{R}$, the inequality $\Phi(x) \ge 1 - \frac{1}{2} \cdot e^{-x^2/2}$ holds (see e.g., (Roch 2014, Equation 2.24)). Applying this, we get

$$\Phi\left(\sqrt{\alpha}\right) \ge 1 - \frac{1}{2} \cdot e^{-\alpha/2}.$$

Chaining the above inequality with $1 - \gamma > \Phi(\sqrt{\alpha})$ and rearranging, we get $\alpha < 2\ln\left(\frac{1}{2\gamma}\right) = -2 \cdot \ln(2\gamma)$. \Box

Lemma 7.10 (Lower bound on change in $\Phi(\sqrt{\cdot})$). For all $0 < \gamma \le 1$, $\Delta_0 > 0$, $\alpha \in (0, -2 \cdot \ln(2\gamma))$, and $\Delta \ge \Delta_0$, it holds that

$$\Phi(\sqrt{\alpha + \Delta}) - \Phi(\sqrt{\alpha}) \ge \frac{2}{\sqrt{\pi}} \cdot \frac{\gamma^{\frac{3}{2}} \cdot \Delta_0}{\sqrt{\frac{2}{e} + \Delta_0} \cdot (2 + \Delta_0)}.$$
(26)

We note that the bound in Lemma 7.10 weakens as Δ_0 (and so, Δ) increases. To see this, observe that the LHS in Equation (26) is an increasing function of Δ . In contrast, if Δ_0 is large enough, the RHS in Equation (26) is a decreasing function of Δ_0 . Nevertheless, Lemma 7.10 suffices to prove Theorem 7.5.

The proof of Lemma 7.10 appears in Section 7.2.7.

7.2.2 Proof of Theorem 7.5

In this section, we present a proof of Theorem 7.5. We begin with the necessary notation and the lemmas (Lemmas 7.11 and 7.12) in Section 7.2.3. Next, in Section 7.2.4, we complete the proof of Theorem 7.5 assuming Lemmas 7.11 and 7.12. Finally, in Section 7.2.5, we present the proofs of Lemmas 7.11 and 7.12 respectively.

Recall that we are given distribution \mathcal{D} which satisfies Definition 7.2. We assume that the statistics returned by the two summary statistic subroutines are exact (Assumption 1).

Fix any iteration $t \in [d']$. Our goals are to prove that for each auxiliary feature $\ell \in [d'] \setminus Q(t)$, AUC_{g(t)}(S(t), Z^{ℓ}) – AUC_{g(t)}(X) $\geq \frac{1}{4} \cdot (\gamma^{(t)})^{3/2} \cdot (\beta_{\ell}^{(t)} \cdot (1 - \delta_{\ell})^{(t)})^2$, and that, in this iteration, fairAUC improves the AUC for the current disadvantaged group g(t), by at least

$$AUC_{g(t)}(X, Z^{i}) - AUC_{g(t)}(X) > \max_{\ell \in [d']} \frac{1}{4} \cdot (\gamma^{(t)})^{3/2} \cdot (\beta_{\ell}^{(t)} \cdot (1 - \delta_{\ell})^{(t)})^{2}.$$

Fix any auxiliary feature $\ell \in [d']$. Then the proof proceeds in two broad steps. First, we show that $AUC(X, Z^{\ell})$ is lower bounded by $AUC(S(t), Z^{\ell})$ (Lemma 7.11). Then, we derive an explicit formula and lower bound for $AUC(S(t), Z^{\ell})$ (Lemma 7.12). This formula is the same as the formula used to compute $AUC_{q(t)}(S(t), Z^{\ell})$ in fairAUC. Thus, fairAUC selects the auxiliary feature *i* where

$$i \in \operatorname{argmax}_{\ell \in [d']} \operatorname{AUC}_{g(t)}(S(t), Z^{\ell})$$

Combining this with a lower bound $AUC(S(t), Z^{\ell})$ for any $\ell \in [d']$, we get that the auxiliary feature *i* selected by fairAUC, satisfies

$$AUC_{g(t)}(S(t), Z^{i}) > \max_{\ell \in [d']} \frac{1}{4} \cdot (\gamma^{(t)})^{3/2} \cdot (\beta_{\ell}^{(t)} \cdot (1 - \delta_{\ell})^{(t)})^{2}.$$

Then, using Lemma 7.11, we get that the auxiliary feature selected by fairAUC improves the AUC for g(t) by at least $\max_{\ell \in [d']} \frac{1}{4} \cdot (\gamma^{(t)})^{3/2} \cdot (\beta_{\ell}^{(t)} \cdot (1 - \delta_{\ell})^{(t)})^2$. Finally since the choice of t was arbitrary, we get that the result holds for all $t \in [d']$.

7.2.3 Additional Notation and Supporting Lemmas

We begin by presenting the two lemmas used in the proof of Theorem 7.5.

Lemma 7.11 (Projection does not increase AUC). Consider three random variables $X \in \mathbb{R}^d$, $Y \in \{0,1\}$, $Z \in \mathbb{R}$, which follow some joint distribution \mathcal{D} . Given $w \in \mathbb{R}^d$, define $S := \langle w, X \rangle$, it holds that

$$\operatorname{AUC}_{\mathcal{D}}(X, Z) \ge \operatorname{AUC}_{\mathcal{D}}(S, Z).$$

The proof of Lemma 7.11 appears in Section 7.2.5.

We require some additional notation to present Lemma 7.12. Fix any iteration $t \in [d']$. Since t will be fixed for the remainder of the proof, we drop the superscripts from $\gamma^{(t)}$, $\Delta v_{\ell}^{(t)}$, $\beta_{\ell}^{(t)}$, and $\delta_{\ell}^{(t)}$. From Definition 7.2, we know that conditioned on A and Y, $X \cup Z$ follows an *m*-variate normal distribution. It follows that X(t) also has a Gaussian distribution conditioned on Y and A (see e.g., (Stirzaker 2003, Theorem 5, Section 8.4)). Suppose for all $y \in \{0, 1\}$

$$X(t) \mid Y = y, A = g(t) \sim \mathcal{N}(\mu_y, \Sigma_y),$$
 (Binormality of acquired features) (27)

and for all $y \in \{0, 1\}$ and $\ell \in [d'] \setminus Q(t)$,

$$Z^{\ell} \mid Y = y, A = g(t) \sim \mathcal{N}(v_{y\ell}, \sigma_{y\ell}^2),$$
 (Binormality of auxiliary features) (28)

where $\mu_0, \mu_1 \in \mathbb{R}^{d+t}, \Sigma_0, \Sigma_1 \in \mathbb{R}^{(d+t) \times (d+t)}$, and for all $\ell \in [d'] \setminus Q(t), v_{0\ell}, v_{1\ell} \in \mathbb{R}$ and $\sigma_{0\ell}^2, \sigma_{1\ell}^2 \ge 0$. Note that $\{Z^\ell\}_{\ell \in [d'] \setminus Q(t)}$ can be correlated with each other (and with X(t)).

Next, we show that $\Sigma_0 + \Sigma_1$ is invertible. Towards this, notice that Definition 7.2 requires that for any $y \in \{0,1\}$ and $g \in \{a,b\}$, conditioned on Y = y and A = g the covariance matrix of all *m* features, M_{yg} ,

is invertible. Since covariance matrices are positive semi-definite (PSD) and any invertible PSD matrix is positive definite (PD), it follows that M_{yg} is PD. Notice that Σ_0 and Σ_1 are submatrices of M_{0g} and M_{1g} . Since submatrices of PD matrices are also PD, it follows that Σ_0 and Σ_1 are PD. Then, $\Sigma_0 + \Sigma_1$ is PD. Thus, $\Sigma_0 + \Sigma_1$ is invertible.

Define $\Delta \mu$ and $\Delta \mu_S$ to be the difference in class-conditional means of X(t) and S respectively

$$\Delta \mu \coloneqq |\mu_1 - \mu_0|,$$

$$\Delta \mu_S \coloneqq |\mathbb{E}[S(t) \mid Y = 1, A = g(t)] - \mathbb{E}[S(t) \mid Y = 0, A = g(t)]|.$$
(29)

Similarly, for all $\ell \in [d']$, define Δv_{ℓ} to be the difference in class-conditional means of Z^{ℓ}

$$\Delta v_{\ell} \coloneqq |v_{1\ell} - v_{0\ell}| \,. \tag{30}$$

For all $y \in \{0, 1\}$, define σ_{yS}^2 to be the variance of S(t) conditioned on Y = y:

$$\sigma_{yS}^2 \coloneqq \operatorname{Var}[S(t) \mid Y = y, A = g(t)]$$

Finally, for all $y \in \{0, 1\}$ and $\ell \in [d']$, define $\rho_{y\ell}$ as the covariance of S(t) and Z^{ℓ} ,

$$\rho_{y\ell} \coloneqq \operatorname{Cov}[S(t), Z^{\ell} \mid Y = y, A = g(t)].$$

Using the definition of S(t) (Equation (15)), we can compute $\Delta \mu_S$ in terms of $\Delta \mu$.

$$\begin{aligned} \Delta\mu_S &:= |\mathbb{E}[S(t) | Y = 1, A = g(t)] - \mathbb{E}[S(t) | Y = 0, A = g(t)]| \\ \stackrel{(15)}{=} |\mathbb{E}\left[\Delta\mu^{\top}(\Sigma_0 + \Sigma_1)^{-1}X(t) | Y = 1, A = g(t)\right] - \mathbb{E}\left[\Delta\mu^{\top}(\Sigma_0 + \Sigma_1)^{-1}X(t) | Y = 0, A = g(t)\right]| \\ &= |\Delta\mu^{\top}(\Sigma_0 + \Sigma_1)^{-1} \left(\mathbb{E}\left[X(t) | Y = 1, A = g(t)\right] - \mathbb{E}\left[X(t) | Y = 1, A = g(t)\right]\right)| \\ \stackrel{(29)}{=} |\Delta\mu^{\top}(\Sigma_0 + \Sigma_1)^{-1}\Delta\mu| \\ &= \Delta\mu^{\top}(\Sigma_0 + \Sigma_1)^{-1}\Delta\mu. \end{aligned}$$
(Using that $(\Sigma_0 + \Sigma_1)^{-1}$ is a PD matrix) (31)

Similarly, for all $y \in \{0, 1\}$, we can also compute σ_{yS}^2 . For y = 1, we have

$$\begin{aligned}
& \stackrel{2}{_{1S}} := \operatorname{Var}[S(t) \mid Y = 1, A = g(t)] \\
& \stackrel{(15)}{=} \operatorname{Var}\left[\Delta\mu^{\top}(\Sigma_{0} + \Sigma_{1})^{-1}X(t) \mid Y = 1, A = g(t)\right] \\
& = \Delta\mu^{\top}(\Sigma_{0} + \Sigma_{1})^{-1}\Sigma_{1}(\Sigma_{0} + \Sigma_{1})\Delta\mu.
\end{aligned}$$
(32)

In the last equality, we use the fact that for any vector $w \in \mathbb{R}^{d+t}$ and random variable $X(t) \in \mathbb{R}^{d+t}$ with covariance matrix $\Sigma \in \mathbb{R}^{(d+t) \times (d+t)}$, it holds that $\operatorname{Var}[\langle w, X(t) \rangle] = w^{\top} \Sigma w$. Similarly, for y = 0 we have that

$$\sigma_{0S}^2 := \Delta \mu^\top (\Sigma_0 + \Sigma_1)^{-1} \Sigma_0 (\Sigma_0 + \Sigma_1) \Delta \mu.$$
(33)

Combining Equations (32) and (33), we get

 σ

$$\sigma_{0S}^2 + \sigma_{1S}^2 \stackrel{(32),(33)}{=} \Delta \mu^\top (\Sigma_0 + \Sigma_1)^{-1} \Delta \mu.$$
(34)

Lemma 7.12. If \mathcal{D} satisfies Equations (27) and (28), then it holds that

$$\operatorname{AUC}_{g(t)}(S(t), Z) \coloneqq \Phi\left(\sqrt{\begin{bmatrix}\Delta\mu_S & \Delta v_\ell\end{bmatrix}} \begin{bmatrix} \sigma_{0S}^2 + \sigma_{1S}^2 & \rho_{0\ell} + \rho_{1\ell} \\ \rho_{0\ell} + \rho_{1\ell} & \sigma_{0\ell}^2 + \sigma_{1\ell}^2 \end{bmatrix}^{-1} \begin{bmatrix}\Delta\mu_S \\ \Delta v_\ell\end{bmatrix}}\right).$$
(35)

Further, let $\alpha \geq 0$, be such that $AUC_{g(t)}(X) = \Phi(\sqrt{\alpha})$, then Equation (35) implies that

$$\operatorname{AUC}_{g(t)}(S(t), Z) \ge \Phi\left(\sqrt{\alpha + \frac{\left(\Delta v_{\ell} - \left(\rho_{0\ell} + \rho_{1\ell}\right)\right)^2}{\sigma_{0\ell}^2 + \sigma_{1\ell}^2}}\right).$$
(36)

The proof of Lemma 7.12 appears in Section 7.2.5. Define $\alpha \geq 0$ to be a constant, such that

$$AUC_{q(t)}(X(t)) = \Phi\left(\sqrt{\alpha}\right). \tag{37}$$

(α is uniquely defined since $\Phi(\sqrt{\cdot})$ is a strictly increasing function.) Further, define $\Delta' \in \mathbb{R}$ to be the term added to α in Equation (36):

$$\Delta' \coloneqq \frac{\left(\Delta v - (\rho_{0\ell} + \rho_{1\ell})\right)^2}{\sigma_{0\ell}^2 + \sigma_{1\ell}^2}.$$
(38)

7.2.4 Proof of Theorem 7.5

Now we are ready to complete the proof of Theorem 7.5.

Proof. Proof of Theorem 7.5 Fix any auxiliary feature $\ell \in [d']$. Consider two cases:

Case A $(|\rho_{0\ell} + \rho_{1\ell}| < \Delta v_{\ell})$: In this case, from Equation (18), we have that $\delta_{\ell} = \Delta v_{\ell}^{-1} \cdot |\rho_{0\ell} + \rho_{1\ell}| \in (0, 1)$. Thus,

$$\Delta' = \frac{\left(\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell})\right)^2}{\sigma_{0\ell}^2 + \sigma_{1\ell}^2} \ge \frac{\Delta v_{\ell}^2 \cdot (1 - \delta_{\ell})^2}{\sigma_{0\ell}^2 + \sigma_{1\ell}^2}.$$

Case B $(|\rho_{0\ell} + \rho_{1\ell}| \ge \Delta v_\ell)$: In this case, from Equation (18), we have that $\delta_\ell = 1$. Thus,

$$\Delta' = \frac{\left(\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell})\right)^2}{\sigma_{0\ell}^2 + \sigma_{1\ell}^2} \ge 0 = \frac{\Delta v_{\ell}^2 \cdot (1 - \delta_{\ell})^2}{\sigma_{0\ell}^2 + \sigma_{1\ell}^2}.$$

Combining both cases and using Equation (17), we can lower bound Δ' (defined in Equation (38)) as follows

$$\Delta' = \frac{\left(\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell})\right)^2}{\sigma_{0\ell}^2 + \sigma_{1\ell}^2} \stackrel{\text{(Cases A and B)}}{\geq} \frac{\Delta v_{\ell}^2 \cdot (1 - \delta_{\ell})^2}{\sigma_{0\ell}^2 + \sigma_{1\ell}^2} \stackrel{(17)}{\geq} \beta_{\ell}^2 \cdot (1 - \delta_{\ell})^2.$$
(39)

Define Δ_0 as the RHS of the above equation, i.e., $\Delta_0 \coloneqq \beta_\ell^2 (1 - \delta_\ell)^2$. Then, we can rewrite Inequality (39) as

$$\Delta' \ge \Delta_0. \tag{40}$$

Using Lemma 7.12, we can show a lower bound on the improvement in the AUC

$$\begin{aligned} \operatorname{AUC}_{g(t)}(S(t), Z^{\ell}) - \operatorname{AUC}_{g(t)}(X(t)) & \stackrel{(37)}{=} & \operatorname{AUC}_{g(t)}(S(t), Z) - \Phi\left(\sqrt{\alpha}\right) \\ & \stackrel{(38), \text{ Lemma 7.12}}{\geq} \Phi\left(\sqrt{\alpha + \Delta'}\right) - \Phi\left(\sqrt{\alpha}\right) \\ & \stackrel{\geq}{\geq} \frac{2}{\sqrt{\pi}} \cdot \frac{\gamma^{\frac{3}{2}} \cdot \Delta_{0}}{\sqrt{(2/e) + \Delta_{0}} \cdot (2 + \Delta_{0})} \\ & \text{ (Using Equation (40), Lemma 7.9, and Lemma 7.10) (41)} \\ & \stackrel{\geq}{\geq} \frac{2}{3\sqrt{\pi}} \cdot \frac{\gamma^{\frac{3}{2}} \cdot \Delta_{0}}{\sqrt{(2/e) + 1}} \\ & \text{ (Using that } 0 \leq \delta_{\ell}, \beta_{\ell} \leq 1 \text{ and, hence, } \Delta_{0} \coloneqq \beta_{\ell}^{2} \cdot (1 - \delta_{\ell})^{2} \leq 1) \\ & \stackrel{\geq}{\geq} \frac{1}{3.51} \cdot \gamma^{\frac{3}{2}} \cdot \Delta_{0} \\ & \stackrel{\leq}{\to} \frac{1}{3.51} \cdot \gamma^{\frac{3}{2}} \cdot \beta_{\ell}^{2} \cdot (1 - \delta_{\ell})^{2} \text{ (Substituting } \Delta_{0} \coloneqq \beta_{\ell}^{2} \cdot (1 - \delta_{\ell})^{2}) \\ & \stackrel{\geq}{\geq} \frac{1}{4} \cdot \gamma^{\frac{3}{2}} \cdot \beta_{\ell}^{2} \cdot (1 - \delta_{\ell})^{2}. \end{aligned}$$

Recall that fairAUC selects an auxiliary feature i, satisfying

$$i \in \operatorname{argmax}_{\ell \in [d']} \operatorname{AUC}_{g(t)}(S(t), Z^{\ell}).$$
 (43)

Using Equations (42) and (43), we get that

$$\operatorname{AUC}_{g(t)}(S(t), Z^{i}) - \operatorname{AUC}_{g(t)}(X(t)) \stackrel{(43)}{=} \max_{\ell \in [d']} \operatorname{AUC}_{g(t)}(S(t), Z^{\ell}) - \operatorname{AUC}_{g(t)}(X(t))$$

$$\stackrel{(42)}{\geq} \max_{\ell \in [d']} \frac{1}{4} \cdot \gamma^{3/2} \beta_{\ell}^{-2} (1 - \delta_{\ell})^{2}. \tag{44}$$

Finally, using Lemma 7.11, we get that

$$\operatorname{AUC}_{g(t)}(X(t), Z^{i}) - \operatorname{AUC}_{g(t)}(X(t)) \stackrel{\operatorname{Lemma \ 7.11}}{\geq} \operatorname{AUC}_{g(t)}(S(t), Z^{i}) - \operatorname{AUC}_{g(t)}(X)$$

$$\stackrel{(44)}{\geq} \max_{\ell \in [d']} \frac{1}{4} \cdot \gamma^{3/2} \beta_{\ell}^{2} (1 - \delta_{\ell})^{2}.$$

7.2.5 Proof of Supporting Lemmas

Proof. Proof of Lemma 7.12 Equation (35) follows by using Lemma 7.7 with d = 2. To see this, note that by Equation (15), S(t) is a fixed projection of the random variable X. Since conditioned on Y and $A, X \cup Z$ is distributed according to a multivariate Gaussian distribution (Definition 7.2), it follows that X(t), and so S(t), also has a Gaussian distribution conditioned on Y and A (see e.g., (Stirzaker 2003, Theorem 5, Section 8.4)). Now Equation (35) follows from Lemma 7.7 by substituting appropriate values for the covariance matrix between S(t) and Z, and the means of S(t) and Z.

Equation (36) follows by expanding Equation (35). Consider the expression inside $\Phi(\sqrt{\cdot})$ in Equation (35). We have

$$\begin{bmatrix} \sigma_{0S}^2 + \sigma_{1S}^2 & \rho_{0\ell} + \rho_{1\ell} \\ (\rho_{0\ell} + \rho_{1\ell}) & \sigma_{0\ell}^2 + \sigma_{1\ell}^2 \end{bmatrix}^{-1} = \frac{1}{(\sigma_{0S}^2 + \sigma_{1S}^2) \cdot (\sigma_{0\ell}^2 + \sigma_{1\ell}^2) - (\rho_{0\ell} + \rho_{1\ell})^2} \begin{bmatrix} \sigma_{0\ell}^2 + \sigma_{1\ell}^2 & -(\rho_{0\ell} + \rho_{1\ell}) \\ -(\rho_{0\ell} + \rho_{1\ell}) & \sigma_{0S}^2 + \sigma_{1S}^2 \end{bmatrix}^{-1}$$

Evaluating the rest of the expression, we have

$$\begin{aligned} \frac{1}{(\sigma_{0S}^{2} + \sigma_{1S}^{2}) \cdot (\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}) - (\rho_{0\ell} + \rho_{1\ell})^{2}} \cdot \left[\Delta \mu_{S} \quad \Delta v_{\ell}\right] \begin{bmatrix} \sigma_{0\ell}^{2} + \sigma_{1\ell}^{2} & -(\rho_{0\ell} + \rho_{1\ell}) \\ -(\rho_{0\ell} + \rho_{1\ell}) & \sigma_{0S}^{2} + \sigma_{1S}^{2} \end{bmatrix} \begin{bmatrix} \Delta \mu_{S} \\ \Delta v_{\ell} \end{bmatrix} \\ &= \frac{1}{(\sigma_{0S}^{2} + \sigma_{1S}^{2}) \cdot (\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}) - (\rho_{0\ell} + \rho_{1\ell})^{2}} \cdot \begin{bmatrix} \Delta \mu_{S} \cdot (\sigma_{\ell\ell}^{2} + \sigma_{1\ell}^{2}) - \Delta v_{\ell} \cdot (\rho_{0\ell} + \rho_{1\ell}) \\ -\Delta \mu_{S} \cdot (\rho_{0\ell} + \rho_{1\ell}) + \Delta v_{\ell} \cdot (\sigma_{0S}^{2} + \sigma_{1S}^{2}) \end{bmatrix}^{\top} \begin{bmatrix} \Delta \mu_{S} \\ \Delta v_{\ell} \end{bmatrix} \\ &= \frac{1}{(\sigma_{0S}^{2} + \sigma_{1S}^{2}) \cdot (\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}) - (\rho_{0\ell} + \rho_{1\ell})^{2}} \cdot (\Delta \mu_{S}^{2} \cdot (\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}) - 2\Delta \mu_{S} \Delta v_{\ell} \cdot (\rho_{0\ell} + \rho_{1\ell}) + \Delta v_{\ell}^{2} \cdot (\sigma_{0S}^{2} + \sigma_{1S}^{2}) \end{bmatrix} \\ \stackrel{(31) = (34)}{=} \frac{1}{\Delta \mu_{S} \cdot (\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}) - (\rho_{0\ell} + \rho_{1\ell})^{2}} \cdot (\Delta \mu_{S}^{2} \cdot (\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}) - 2\Delta \mu_{S} \cdot \Delta v_{\ell} \cdot (\rho_{0\ell} + \rho_{1\ell}) + \Delta v_{\ell}^{2} \cdot \Delta \mu_{S}) \\ \stackrel{(\Delta \mu_{S} \geq 0)}{=} \frac{1}{(\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}) - \frac{(\rho_{0\ell} + \rho_{1\ell})^{2}}{\Delta \mu_{S}}} \cdot (\Delta \mu_{S} \cdot (\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}) - 2\Delta v_{\ell} \cdot (\rho_{0\ell} + \rho_{1\ell}) + \Delta v_{\ell}^{2}) \\ &= \Delta \mu_{S} + \frac{1}{(\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}) - \frac{(\rho_{0\ell} + \rho_{1\ell})^{2}}{\Delta \mu_{S}}} \cdot (\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell}))^{2} \\ &\geq \Delta \mu_{S} + \frac{(\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell}))^{2}}{\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}}} \cdot (\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell}))^{2} \\ &\geq \Delta \mu_{S} + \frac{(\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell}))^{2}}{\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}}} \cdot (\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell}))^{2} \\ &\geq \Delta \mu_{S} + \frac{(\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell}))^{2}}{\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}}} \cdot (\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell}))^{2} \\ &\leq \Delta \mu_{S} + \frac{(\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell}))^{2}}{\sigma_{0\ell}^{2} + \sigma_{1\ell}^{2}}} \cdot (\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell}))^{2}} \\ &\leq \Delta \mu_{S} + \frac{(\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell})^{2}}{\sigma_{\ell}^{2} + \sigma_{1\ell}^{2}}} \cdot (\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell}))^{2}} \\ &\leq \Delta \mu_{S} + \frac{(\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell})^{2}}{\sigma_{\ell}^{2} + \sigma_{1\ell}^{2}}} \cdot (\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell})^{2}} \\ &\leq \Delta \mu_{S} + \frac{(\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell})^{2}}{\sigma_{\ell}^{2} + \sigma_{1\ell}^{2}}} \cdot (\Delta v_{\ell} - (\rho_{0\ell} + \rho_{1\ell})^{2}} + \Delta v_{\ell}^{2} + \Delta v_{\ell}^{2}} + \Delta v_{\ell}^{2} + \Delta v_{\ell}^{2} + \Delta v$$

Substituting Equation (45) in Lemma 7.7, and using the fact that $\Phi(\sqrt{\cdot})$ is an increasing function, we have

$$\operatorname{AUC}_{g(t),\mathcal{D}}(S(t),Z) \ge \Phi\left(\sqrt{\Delta\mu^{\top}(\Sigma_0 + \Sigma_1)^{-1}\Delta\mu + \frac{\left(\Delta v_{\ell} - \left(\rho_{0\ell} + \rho_{1\ell}\right)\right)^2}{\sigma_{0\ell}^2 + \sigma_{1\ell}^2}}\right).$$
(46)

From Lemma 7.7, we also have that

$$\operatorname{AUC}_{g(t),\mathcal{D}}(X(t)) = \Phi\left(\sqrt{\Delta\mu^{\top}(\Sigma_0 + \Sigma_1)^{-1}\Delta\mu}\right).$$

Thus, $\alpha = \Delta \mu^{\top} (\Sigma_0 + \Sigma_1)^{-1} \Delta \mu$. Combining this with Equation (46), we get

$$\operatorname{AUC}_{g(t)}(S(t), Z) \ge \Phi\left(\sqrt{\alpha + \frac{\left(\Delta v_{\ell} - \left(\rho_{0\ell} + \rho_{1\ell}\right)\right)^2}{\sigma_{0\ell}^2 + \sigma_{1\ell}^2}}\right).$$

Proof. Proof of Lemma 7.11 By definition of the AUC (Definition 7.4), we have

$$AUC(X,Z) \coloneqq \max_{v \in \mathbb{R}^{d+1}, \psi} AUC(v,\psi,X,Z) \quad \text{and} \quad AUC(S,Z) \coloneqq \max_{v \in \mathbb{R}^2, \psi} AUC(v,\psi,S,Z).$$
(47)

As shown in the proof of Lemma 7.7, a generalized linear model G defined by weight $w \in \mathbb{R}^d$, increasing and invertible link function $\psi \colon \mathbb{R} \to \mathbb{R}$, and threshold $t \in \mathbb{R}$, makes the same predictions as the linear classifier C defined with weights w and threshold $\psi(t)$. In particular, G and C have the same AUC. Combining this with Equation (47), it follows that:

$$AUC(X,Z) \coloneqq \max_{v \in \mathbb{R}^{d+1}} AUC(v,I,\psi,X,Z) \quad \text{and} \quad AUC(S,Z) \coloneqq \max_{v \in \mathbb{R}^2} AUC(v,I,\psi,S,Z).$$
(48)

where $I \colon \mathbb{R} \to \mathbb{R}$ is the identity function. Define vectors

$$v_{2} \coloneqq \operatorname*{argmax}_{v \in \mathbb{R}^{2}} \operatorname{AUC}(v, I, S, Z), \quad \text{and} \quad v_{1} \coloneqq \begin{bmatrix} w & 0\\ 0 & 1 \end{bmatrix} v_{2} \in \mathbb{R}^{d+1}.$$
(49)

Notice that the linear classifier using v_1 on X and Z, is identical to the linear classifier using v_2 on S and Z:

$$\langle v_1, (X^1, \dots, X^d, Z) \rangle \stackrel{(49)}{=} v_2^\top \begin{bmatrix} w^\top & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Z \end{bmatrix} \stackrel{(S:=\langle w, X \rangle)}{=} v_2^\top \begin{bmatrix} S \\ Z \end{bmatrix}.$$
 (50)

Using this, we have

$$AUC(X,Z) \stackrel{(48)}{\geq} AUC(v_1,I,X,Z) \stackrel{(50)}{=} AUC(v_2,I,S,Z) \stackrel{(49)}{=} AUC(S,Z).$$

7.2.6 Proof of Theorem 7.6

Proof. Proof of Theorem 7.6 The proof of Theorem 7.6 follows from Equation (42) and Lemma 7.11 in the proof of Theorem 7.5. In the proof of Theorem 7.5 only Equation (43) uses the fact that g(t) is the disadvantaged group in iteration t. In particular, the proof of Equation (42) (which occurs before Equation (43)) does not use the fact that g(t) is the disadvantaged group in iteration t. Thus, we can repeat the proof of Equation (42) by substituting g(t) with $\hat{g}(t)$. Then for all $\ell \in [d'] \setminus Q(t)$, it holds that²¹

$$\operatorname{AUC}_{\hat{g}(t)}(S(t), Z^{\ell}) - \operatorname{AUC}_{\hat{g}(t)}(X(t)) \ge \frac{1}{4} \cdot \left(\hat{\gamma}^{(t)}\right)^{3/2} \cdot \left(\hat{\beta}_{\ell}^{(t)}(1 - \hat{\delta}_{\ell}^{(t)})\right)^{2}.$$
(51)

²¹Recall that in the proof of Theorem 7.5, we dropped the superscript on $\hat{\gamma}^{(t)}$, $\hat{\beta}^{(t)}_{\ell}$, and $\hat{\delta}^{(t)}_{\ell}$. Here, we return to specifying the superscripts.

The proof of Lemma 7.11 does not refer to g(t). Thus, we can use it directly. This gives us that for all $\ell \in [d'] \setminus Q(t)$

$$\operatorname{AUC}_{\hat{g}(t)}(X(t), Z^{\ell}) - \operatorname{AUC}_{\hat{g}(t)}(X(t)) \stackrel{\text{Lemma 7.11}}{\geq} \operatorname{AUC}_{\hat{g}(t)}(S(t), Z^{\ell}) - \operatorname{AUC}_{\hat{g}(t)}(X) \\ \stackrel{(51)}{\geq} \quad \frac{1}{4} \cdot \left(\hat{\gamma}^{(t)}\right)^{3/2} \cdot \left(\hat{\beta}_{\ell}^{(t)}(1 - \hat{\delta}_{\ell}^{(t)})\right)^{2}.$$

In particular, this holds for the feature $i \in [d'] \setminus Q(t)$, selected by fairAUC.

7.2.7 Proof of Lemma 7.10

Proof. Proof of Lemma 7.10.

$$\Phi(\sqrt{\alpha + \Delta}) - \Phi(\sqrt{\alpha}) = \int_{\sqrt{\alpha}}^{\sqrt{\alpha + \Delta}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$
$$\geq \int_{\sqrt{\alpha}}^{\sqrt{\alpha + \Delta_0}} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$

(Using that $\Delta \geq \Delta_0$ and that the RHS is an increasing function of Δ)

$$\geq \int_{\sqrt{\alpha}}^{\sqrt{\alpha}+\Delta_0} \frac{y}{\sqrt{\alpha}+\Delta_0} \frac{1}{\sqrt{2\pi}} e^{-y^2/2} dy$$
 (Using the fact that for all $y \in [\sqrt{\alpha}, \sqrt{\alpha}+\Delta_0], \ \frac{y}{\sqrt{\alpha}+\Delta_0} \leq 1$)

$$= \frac{-e^{-y^2/2}}{\sqrt{\alpha + \Delta_0}} \cdot \frac{1}{\sqrt{2\pi}} \Big|_{\sqrt{\alpha}}^{\sqrt{\alpha + \Delta_0}}$$
$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{\alpha}{2}} \cdot \left(1 - e^{-\Delta_0/2}\right)}{\sqrt{\alpha + \Delta_0}}$$
$$\geq \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{3\alpha}{2}} \cdot \left(1 - e^{-\Delta_0/2}\right)}{\sqrt{\frac{2}{e} + \Delta_0}}$$

(Using that for all $\alpha, \Delta_0 > 0, \frac{e^{-\alpha/2}}{\sqrt{\alpha + \Delta_0}} \ge \frac{e^{-3\alpha/2}}{\sqrt{\frac{2}{e} + \Delta_0}}$; see Equation (53)) (52)

$$\geq \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-\frac{3\alpha}{2}} \cdot \Delta_0}{\sqrt{\frac{2}{e} + \Delta_0} \cdot (2 + \Delta_0)}$$
(Using the fact that for all $x \in \mathbb{R}, 1 - e^{-x/2} \geq \frac{x}{2+x}$)

$$\geq \frac{1}{\sqrt{2\pi}} \cdot \frac{(2\gamma)^{\frac{3}{2}} \cdot \Delta_0}{\sqrt{\frac{2}{e} + \Delta_0} \cdot (2 + \Delta_0)}$$

(Using that $e^{-\frac{3\alpha}{2}}$ is a decreasing function of α and $\alpha < -2 \cdot \ln(2\gamma)$)

$$\frac{2}{\sqrt{\pi}} \cdot \frac{\gamma^{\frac{3}{2}} \cdot \Delta_0}{\sqrt{\frac{2}{e} + \Delta_0} \cdot (2 + \Delta_0)}.$$

It remains to prove the following inequality, which was used in Equation (52),

=

$$\forall x, y > 0, \quad \frac{e^{-\frac{x}{2}}}{\sqrt{x+y}} \ge \frac{e^{-\frac{3x}{2}}}{\sqrt{\frac{2}{e}+y}}.$$
 (53)

Using the fact that for all $x \ge 0, e^{-\frac{x}{2}}x \le \frac{2}{e}$, we have that for any x, y > 0

$$e^{-\frac{x}{2}} \cdot (x+y) \le \frac{2}{e} + e^{-\frac{x}{2}}y \stackrel{x>0}{<} \frac{2}{e} + y.$$
 (54)

Hence, we have

$$\frac{e^{-\frac{x}{2}}}{\sqrt{x+y}} = \frac{e^{-\frac{x}{2}} \cdot \sqrt{e^{-\frac{x}{2}}}}{\sqrt{e^{-\frac{x}{2}} \cdot (x+y)}} \stackrel{(54)}{\ge} \frac{e^{-\frac{3x}{2}}}{\sqrt{\frac{2}{e}+y}}.$$

7.3 Proof of Theoretical Guarantees of noisy fairAUC

In this section, we formally describe the noisy fairAUC procedure, state its theoretical guarantees and prove them.

fairAUC's objective is to increase the AUC of the lowest AUC group. However, fairAUC can increase bias with the acquisition of a new feature. Suppose we have data X and we acquire feature Z so that our new data is X' := (X, Z). We can then have Bias(X') > Bias(X), where Bias(X) and Bias(X') are the values of bias obtained using features X and X' respectively. In this section, we provide an approach that ensures that, in each iteration of fairAUC, the bias is smaller than or equal to the bias in the previous iteration. Thus, overall we can guarantee that the bias will only decrease over rounds as we acquire more features. At a high level, the strategy involves adding a noisy version of Z to the group with higher AUC rather than Z itself. This strategy, however, trades off predictive accuracy to ensure bias does not increase.

To formalize this, consider the t-th iteration of the fairAUC procedure. Let $0 \leq \text{Bias}(t) \leq 1$ be the bias at the *start* of the t-th iteration

$$\mathsf{Bias}(t) \coloneqq 1 - \max\left\{\frac{\mathsf{AUC}_a(X(t))}{\mathsf{AUC}_b(X(t))}, \frac{\mathsf{AUC}_b(X(t))}{\mathsf{AUC}_a(X(t))}\right\}$$

Here, $AUC_g(X)$ denotes the largest AUC, on group $g \in \{a, b\}$, of a generalized linear model that uses X(t) to predict Y (as defined in Definition 7.4). Suppose that in this iteration, fairAUC acquires an auxiliary feature Z. Let Bias(t + 1) be the bias after acquiring Z

$$\mathsf{Bias}(t+1) \coloneqq 1 - \max\left\{\frac{\mathrm{AUC}_a(X(t), Z)}{\mathrm{AUC}_b(X(t), Z)}, \frac{\mathrm{AUC}_b(X(t), Z)}{\mathrm{AUC}_a(X(t), Z)}\right\}$$

While fairAUC does not decrease the AUC of either group (Theorems 7.5 and 7.6), it can increase the bias. We can then have Bias(t + 1) > Bias(t). To avoid this, one can replace Z with its "noisy version," which introduces noise to the group with the higher AUC (where the AUC refers to the measure *after* Z is added). We will show that this ensures that the bias never increases.

Let g(t) be the advantaged group at the *end* of the *t*-th iteration, i.e., the group with the higher AUC. Let $0 \le \lambda \le 1$ be a parameter controlling the extent of the noise. The noisy version of Z is

$$Z_{\lambda} \coloneqq \begin{cases} \lambda \cdot Z + (1 - \lambda) \cdot N & \text{if } A = g(t), \\ Z & \text{if } A \neq g(t). \end{cases}$$
(55)

where N is a standard normal random variable, independent of all d features, the class label Y, and protected group A. In other words, Z_{λ} is constructed from Z by scaling Z by a factor of λ for all samples in the advantaged group and then adding standard normal noise (scaled by $1 - \lambda$) to these samples. Let $\text{Bias}_{\lambda}(t+1)$ be the bias obtained by using features X(t) and Z_{λ}

$$\mathsf{Bias}_{\lambda}(t+1) \coloneqq 1 - \max\left\{\frac{\mathrm{AUC}_{a}(X(t), Z_{\lambda})}{\mathrm{AUC}_{b}(X(t), Z_{\lambda})}, \frac{\mathrm{AUC}_{b}(X(t), Z_{\lambda})}{\mathrm{AUC}_{a}(X(t), Z_{\lambda})}\right\}$$

We show that one can choose a value of λ such that $\text{Bias}_{\lambda}(t+1) \leq \text{Bias}(t)$ and the AUC of neither group decreases compared to their value *before* the iteration. Formally, we prove the following theorem.

Theorem 7.13. Suppose that the *m* features $X \cup Z$, class label Y, and protected group A follow the binormal framework (Definition 7.2). Further, assume that two summary statistics subroutines satisfy Assumption 1.

Then, for each iteration $t \in [d']$ and each auxiliary feature Z, there is a value of $0 \le \lambda \le 1$ such that the bias obtained by using features $(X(t), Z_{\lambda})$ is at most Bias(t),

$$\operatorname{Bias}_{\lambda}(t+1) \leq \operatorname{Bias}(t),$$

and the AUC of neither group decrease compared to their values before the t-th iteration,

$$AUC_a(X(t), Z_\lambda) \ge AUC_a(X(t)) \quad and \quad AUC_b(X(t), Z_\lambda) \ge AUC_b(X(t)).$$
(56)

Thus, Theorem 7.13 shows that adding a suitable amount of noise to the samples in the advantaged group ensures that the selected feature does not increase the amount of bias. At the same time, the AUC improvement for the disadvantaged group is unaffected because we do not add noise to the samples in the disadvantaged group. The AUC improvement for the advantaged group can decrease but we can lower bound the amount of decrease as explained in the following remark.

Remark 7.14. Theorem 7.6's proof lower bounds the increase in the AUC of the advantaged group due to the acquisition of any feature that follows the normal distribution conditioned on the class and the protected group labels. (This, then, directly implies the lower bound in Theorem 7.6.) The same lower bound also applies to the "noisy" feature Z_{λ} because it follows the normal distribution for any class and group. The latter is true because Z_{λ} is a sum of two normally-distributed variables (the selected feature Z and the standard-normal noise) and because the sum of any two normally-distributed variables is another normally-distributed variable. The lower bound depends on Z_{λ} 's moments and correlation with the scores S via parameters that are analogous to those in Theorem 7.6. If the class-wise variances of Z sum to at least α , then the AUC-improvement with the noisy feature is at least $\frac{1}{1+\frac{2(1-\lambda)^2}{\alpha}}$ times the lower bound in Theorem 7.6.

Moreover, even when we add noise to the acquired feature in this iteration. The lower bound in Proposition 4.1 (and that in Theorem 7.6) continues to hold in subsequent iterations. This is because their proofs hold whenever all features (acquired and auxiliary) are normally-distributed, conditioned on class and group and, as we saw in the above remark, if Z is normally-distributed (for each class and group), then so is Z_{λ} .

We also prove two corollaries of Theorem 7.13, which generalize it (Corollary 7.15) and give a stronger result if when an additional condition is satisfied (Corollary 7.16).

Corollary 7.15. Under the same assumptions as Theorem 7.13, for each iteration $t \in [d']$ and each auxiliary feature Z, there is a $\Delta \geq 0$ such that for any value²²

$$Bias(t) - \Delta \le v \le Bias(t+1)$$

there exists a $0 \le \lambda \le 1$ such that $\text{Bias}_{\lambda}(t+1) = v$ and Equation (56) holds.

In the next corollary, note that we can swap groups a and b.

Corollary 7.16. Under the same assumptions as Theorem 7.13, for each iteration $t \in [d']$ and each auxiliary feature Z, if group a is the advantaged group at the end of the t-th iteration (i.e., $AUC_a(X(t), Z) \ge AUC_b(X(t), Z)$) and $AUC_a(X(t)) \le AUC_b(X(t), Z)$, then there exists a value $0 \le \lambda \le 1$ such that $Bias_{\lambda}(t + 1) = 0$ and Equation (56) holds.

Finally, we show that the above method of adding noise to only one group dominates a method that adds noise to both groups. Given parameters $0 \le \alpha, \beta \le 1$, determining the extent of noise on groups a and b, the new method uses

$$Z_{\alpha,\beta} \coloneqq \begin{cases} \alpha \cdot Z + (1-\alpha) \cdot N & \text{if } A = a, \\ \beta \cdot Z + (1-\beta) \cdot N & \text{if } A = b. \end{cases}$$
(57)

Let $\operatorname{Bias}_{\alpha,\beta}(t+1)$ be the bias obtained by using the features $(X(t), Z_{\alpha,\beta})$. We show that for any fixed value of bias, i.e., $\operatorname{Bias}_{\alpha,\beta}(t+1) = \operatorname{Bias}_{\lambda}(t+1)$, the group-wise AUCs obtained by adding noise to only the advantaged group (i.e., using Z_{λ}) weakly Pareto dominates the group-wise AUCs obtained by adding noise to both groups (i.e., using $Z_{\alpha,\beta}$).

²²Note the result is vacuously true if $Bias(t) - \Delta > Bias(t+1)$.

Theorem 7.17. Under the same assumptions as Theorem 7.13, for any $0 \le \alpha, \beta \le 1$ such that $\text{Bias}_{\alpha,\beta}(t+1) \le \text{Bias}(t+1)$, there exists $0 \le \lambda \le 1$ such that $\text{Bias}_{\alpha,\beta}(t+1) = \text{Bias}_{\lambda}(t+1)$ and

 $\operatorname{AUC}_a(X(t), Z_{\lambda}) \ge \operatorname{AUC}_a(X(t), Z_{\alpha,\beta}) \quad and \quad \operatorname{AUC}_b(X(t), Z_{\lambda}) \ge \operatorname{AUC}_b(X(t), Z_{\alpha,\beta}).$

7.3.1 Proof of Theorem 7.13

In this section, we prove Theorem 7.13. We need to show that there exists a $0 \le \lambda \le 1$ such that $\text{Bias}_{\lambda}(t+1) \le \text{Bias}(t)$. Recall that $\text{Bias}_{\lambda}(t+1)$ is a function of $\text{AUC}_a(X(t), Z_{\lambda})$ and $\text{AUC}_b(X(t), Z_{\lambda})$. We will express $\text{AUC}_a(X(t), Z_{\lambda})$ and $\text{AUC}_b(X(t), Z_{\lambda})$ as functions of λ . The proof follows by analyzing these functions. Without loss of generality assume that g(t) = a, i.e.,

$$\operatorname{AUC}_a(X(t), Z) \ge \operatorname{AUC}_b(X(t), Z)$$

Expression for AUC_b($\mathbf{X}(\mathbf{t}), \mathbf{Z}_{\lambda}$). Since $(Z_{\lambda} = Z) \mid A = b$ (Equation (55)), we get that

$$\forall \lambda \in [0,1], \quad \text{AUC}_b(X(t), Z_\lambda) = \text{AUC}_b(X(t), Z).$$
(58)

Hence, $AUC_b(X(t), Z_\lambda)$ is invariant of λ .

Expression for AUC_a(**X**(**t**), \mathbf{Z}_{λ}). Using Lemma 7.7, we can express AUC_a($X(t), Z_{\lambda}$) as a function of the conditional means and covariances of ($X(t), Z_{\lambda}$); conditioned on the events (A = a, Y = 0) or (A = a, Y = 1). We need some additional notation to state this expression. For each $y \in \{0, 1\}$, let the mean and covariance matrix of (X(t), Z) conditioned on (Y = y, A = a) be:

$$\begin{bmatrix} \mu_{X,y} \\ \mu_{Z,y} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Sigma_y & \rho_y \\ \rho_y^{\top} & \sigma_y \end{bmatrix},$$
(59)

where for each $y \in \{0, 1\}$

- $\mu_{X,y} \in \mathbb{R}^{d+t}$ is the mean of X(t)|Y = y, A = a,
- $\mu_{Z,y} \in \mathbb{R}$ is the mean of Z|Y = y, A = a,
- $\Sigma_y \in \mathbb{R}^{(d+t) \times (d+t)}$ is the covariance matrix of X(t)|Y = y, A = a,
- $\sigma_y \in \mathbb{R}$ is the variance of Z|Y = y, A = a, and
- $\rho_y \in \mathbb{R}^{d+t}$ is the covariance of X(t)|Y = y, A = a and Z|Y = y, A = a.

From the definition of Z_{λ} (Equation (55)), Equation (59), and the fact that N is a standard normal variable independent of (X(t), Z, A, Y), we get that the mean and covariance matrix of $(X(t), Z_{\lambda})$ conditioned on (Y = y, A = a) to be:

$$\begin{bmatrix} \mu_{X,y} \\ \lambda \cdot \mu_{Z,y} \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} \Sigma_y & \lambda \cdot \rho_y \\ \lambda \cdot \rho_y^\top & \lambda^2 \cdot \sigma_y + (1-\lambda)^2 \end{bmatrix}.$$
(60)

To simplify the notation, define

$$\Sigma \coloneqq \Sigma_0 + \Sigma_1, \qquad \sigma \coloneqq \sigma_0 + \sigma_1, \qquad \rho \coloneqq \rho_0 + \rho_1,$$
$$\Delta \mu_X \coloneqq |\mu_{X,1} - \mu_{X,0}|, \quad \Delta \mu_Z \coloneqq |\mu_{Z,1} - \mu_{Z,0}|.$$

Substituting the above definitions and Equation (60) in Lemma 7.7, we get that

$$\operatorname{AUC}_{a}(X(t), Z_{\lambda}) = \Phi\left(\begin{bmatrix}\Delta\mu_{X}\\\lambda\cdot\Delta\mu_{Z}\end{bmatrix}^{\top}\begin{bmatrix}\Sigma&\lambda\cdot\rho\\\lambda\cdot\rho^{\top}&\lambda^{2}\cdot\sigma+(1-\lambda)^{2}\end{bmatrix}^{-1}\begin{bmatrix}\Delta\mu_{X}\\\lambda\cdot\Delta\mu_{Z}\end{bmatrix}\right).$$
(61)

Substituting $\lambda = 0$ and $\lambda = 1$ in Equation (61), we recover that

$$AUC_a(X(t), Z_0) = AUC_a(X(t)) \quad and \quad AUC_a(X(t), Z_1) = AUC_a(X(t), Z).$$
(62)

This is expected because $Z_0|g = a$ is an independent standard normal variable, and therefore does not provide "any information" about Y, and because $Z_1 = Z$, respectively. Using Equation (61) we can also prove the following result.

Lemma 7.18. AUC_a($X(t), Z_{\lambda}$) is a continuous function of λ over [0, 1].

Proof of Theorem 7.13. If $AUC_a(X(t), Z) = AUC_b(X(t), Z)$, then Bias(t + 1) = 0. Since $Bias(t) \ge 0$ and $Bias(t + 1) = Bias_1(t + 1)$, we are done. Henceforth, assume that

$$AUC_a(X(t), Z) > AUC_b(X(t), Z).$$
(63)

Case A (AUC_b(X(t), Z) \geq AUC_a(X(t)): In this case

$$\operatorname{AUC}_{a}(X(t), Z_{0}) \stackrel{(62)}{=} \operatorname{AUC}_{a}(X(t))$$

$$\leq \operatorname{AUC}_{b}(X(t), Z)$$

$$\stackrel{(63)}{<} \operatorname{AUC}_{a}(X(t), Z)$$

$$\stackrel{(62)}{=} \operatorname{AUC}_{a}(X(t), Z_{1}).$$

Let $f(\lambda) \coloneqq \operatorname{AUC}_a(X(t), Z_{\lambda})$. f is continuous over [0, 1] by Lemma 7.18. Since f is continuous over [0, 1]and $f(0) \leq \operatorname{AUC}_b(X(t), Z) < f(1)$, by the intermediate value theorem there exists an $\alpha \in [0, 1]$ such that $f(\alpha) = \operatorname{AUC}_b(X(t), Z)$. Using $f(\alpha) = \operatorname{AUC}_b(X(t), Z)$ and Equation (58), we get

$$\mathsf{Bias}_{\alpha}(t+1) = 1 - \frac{\mathrm{AUC}_b(X(t), Z_{\alpha})}{\mathrm{AUC}_a(X(t), Z_{\alpha})} = 1 - \frac{\mathrm{AUC}_b(X(t), Z)}{\mathrm{AUC}_b(X(t), Z)} = 0$$

Hence, in this case Theorem 7.13 follows for $\lambda = \alpha$. This also proves Corollary 7.16.

Case B (AUC_b(X(t), Z) < AUC_a(X(t)): Define

$$\zeta \coloneqq \operatorname{AUC}_b(X(t), Z) - \operatorname{AUC}_b(X(t)).$$
(64)

We have that

$$\operatorname{AUC}_{a}(X(t), Z_{0}) \stackrel{(62)}{=} \operatorname{AUC}_{a}(X(t)) > \operatorname{AUC}_{b}(X(t), Z) \stackrel{(58)}{=} \operatorname{AUC}_{b}(X(t), Z_{0}).$$

Using this inequality, we can express $Bias_0(t+1)$ as

$$\begin{aligned} \mathsf{Bias}_0(t+1) &= 1 - \frac{\mathrm{AUC}_b(X(t), Z_0)}{\mathrm{AUC}_a(X(t), Z_0)} \\ \stackrel{(64)}{=} 1 - \frac{\mathrm{AUC}_b(X(t)) + \zeta}{\mathrm{AUC}_a(X(t), Z_0)} \\ \stackrel{(62)}{=} 1 - \frac{\mathrm{AUC}_b(X(t)) + \zeta}{\mathrm{AUC}_a(X(t))} \end{aligned}$$

Using Lemma 7.11, we have that $AUC_b(X(t), Z) \ge AUC_b(X(t))$. Combining this inequality with the fact that in this case $AUC_b(X(t), Z) < AUC_a(X(t))$, it follows that $AUC_a(X(t)) > AUC_b(X(t))$. Consequently

$$\begin{aligned} \mathsf{Bias}_0(t+1) &= 1 - \max\left\{\frac{\mathrm{AUC}_b(X(t))}{\mathrm{AUC}_a(X(t))}, \frac{\mathrm{AUC}_a(X(t))}{\mathrm{AUC}_b(X(t))}\right\} - \frac{\zeta}{\mathrm{AUC}_a(X(t))} \\ &= \mathrm{Bias}(t) - \frac{\zeta}{\mathrm{AUC}_a(X(t))}. \end{aligned}$$

By Theorems 7.5 and 7.6 and the definition of ζ , we have that $\zeta \ge 0$ and, hence, $\frac{\zeta}{AUC_a(X(t))} \ge 0$. Therefore, in this case, Theorem 7.13 follows for $\lambda = 0$.

Proof of Corollary 7.15.

Proof. Proof of Corollary 7.15 Let $\Delta \coloneqq \min\left\{\frac{\zeta}{\operatorname{AUC}_a(X(t))}, \operatorname{Bias}(t)\right\}$, where ζ is as defined in Equation (64). If $\operatorname{AUC}_a(X(t), Z) = \operatorname{AUC}_b(X(t), Z)$, then $\operatorname{Bias}(t+1) = 0$. Since $\operatorname{Bias}(t) - \Delta \ge 0$, we are done. Henceforth, assume that $\operatorname{AUC}_a(X(t), Z) > \operatorname{AUC}_b(X(t), Z)$. Using Lemma 7.18, Equation (58), and the definition of $\operatorname{Bias}_\lambda(t+1)$, it follows that $\operatorname{Bias}_\lambda(t+1)$ is a continuous function of λ over [0, 1]. **Case A** (AUC_b(X(t), Z) \geq AUC_a(X(t)): In the proof of Theorem 7.13 we showed that there exists a $\alpha \in [0,1]$ such that $\text{Bias}_{\alpha}(t+1) = 0$. Since $\text{Bias}_{\alpha}(t+1) = 0 \leq \text{Bias}(t) - \Delta$ and $\text{Bias}_1(t+1) = \text{Bias}(t+1)$, using the intermediate value theorem it follows that for any $\text{Bias}(t) - \Delta < v \leq \text{Bias}(t+1)$, there exists a $w \in [\alpha, 1]$ such that $\text{Bias}_w(t+1) = v$.

Case B (AUC_b(X(t), Z) < AUC_a(X(t)): In the proof of Theorem 7.13 we showed that Bias₀(t + 1) \leq Bias(t) – Δ . Combining this with the fact that Bias₁(t + 1) = Bias(t + 1), and using the intermediate value theorem it follows that for any Bias(t) – Δ < $v \leq$ Bias(t + 1), there exists a $w \in [0, 1]$ such that Bias_w(t + 1) = v.

Proof of Lemma 7.18

Proof. Proof of Lemma 7.18 Consider the matrix

$$\Lambda(\lambda) \coloneqq \begin{bmatrix} \Sigma & \lambda \cdot \rho \\ \lambda \cdot \rho^\top & \lambda^2 \cdot \sigma + (1 - \lambda)^2 \end{bmatrix}.$$

We claim that $\Lambda(\lambda)$ is invertible for all $0 \le \lambda \le 1$. Suppose this claim is true. Then the lemma follows from Equation (61), because

- $\Phi \colon \mathbb{R} \to \mathbb{R}$ is a continuous function,
- $A \to A^{-1}$ is a continuous function over the set of all invertible matrices.

One can show this, e.g., using (Munkres 2000, Theorem 18.2).

It remains to prove the claim. Recall that in the binormal framework (Definition 7.2), we assume that for each $y \in \{0, 1\}$ and A = a, Σ_y and $\begin{bmatrix} \Sigma_y & \rho_y \\ \rho_y^\top & \sigma_y \end{bmatrix}$ are invertible. Since these matrices are also positive semi-definite for each $y \in \{0, 1\}$, it implies that Σ and $\begin{bmatrix} \Sigma & \rho \\ \rho^\top & \sigma \end{bmatrix}$ are invertible. (This is proved in the paragraph below Equation (28)). This proves that $\Lambda(\lambda)$ is invertible for $\lambda \in \{0, 1\}$. Further, using that $\Lambda(1)$ and Σ are invertible and (Lu and Shiou 2002, Theorem 2.1(i)), we also get that $\frac{1}{\sigma - \rho^\top \Sigma^{-1} \rho} \neq 0$. Further, since $\Lambda(1)$ is positive definite, it must be that $\frac{1}{\sigma - \rho^\top \Sigma^{-1} \rho} > 0$ or equivalently

$$\sigma - \rho^{\top} \Sigma^{-1} \rho > 0. \tag{65}$$

This shows that for any $0 < \lambda < 1$

$$\lambda^{2} \cdot (\sigma + \rho^{\top} \Sigma^{-1} \rho) + (1 - \lambda)^{2} > (1 - \lambda)^{2} \qquad (\text{Using Equation (65) and } \lambda \neq 0) \\ > 0. \qquad (\text{Using } 0 < \lambda < 1)$$

Now (Lu and Shiou 2002, Theorem 2.1(i)) implies that $\Lambda(\lambda)$ is invertible for all $0 < \lambda < 1$.

7.3.2 Proof of Theorem 7.17

Proof. Proof of Theorem 7.17 To show the existence of the claimed λ , it suffices to show that $\mathsf{Bias}_{\alpha,\beta}(t+1) \geq \mathsf{Bias}(t) - \Delta$ and use Corollary 7.15. In the proof of Corollary 7.15, we set

$$\Delta \coloneqq \min\left\{\frac{\operatorname{AUC}_b(X(t), Z) - \operatorname{AUC}_b(X(t))}{\operatorname{AUC}_a(X(t))}, \operatorname{Bias}(t)\right\}.$$
(66)

Case A (AUC_b(X(t), Z) \geq AUC_a(X(t))): In this case, Bias(t) = $1 - \frac{AUC_b(X(t))}{AUC_a(X(t))}$. Using this and the definition of Δ , one can verify that in this case $\Delta = \text{Bias}(t)$. Hence, $\text{Bias}_{\alpha,\beta}(t+1) \geq 0 = \text{Bias}(t) - \Delta$.

Case B (AUC_b(X(t), Z) < AUC_a(X(t))): We will use the following lemma:

Lemma 7.19. For all $0 \le \alpha, \beta \le 1$ and both groups $g \in \{a, b\}$, $AUC_g(X(t)) \le AUC_g(X(t), Z_{\alpha,\beta}) \le AUC_g(X(t), Z)$.

Proof. Proof of Lemma 7.19 Note that conditioned on A = g, $Z_{\alpha,\beta}$ is a linear combination of Z and N. Hence, $\operatorname{AUC}_g(X, Z_{\alpha,\beta}) \leq \operatorname{AUC}_g(X, Z, N)$ (see Lemma 7.11). Further, since N is independent of Y, X, Z, and A, using Lemma 7.7 it follows that $\operatorname{AUC}_g(X, Z, N) = \operatorname{AUC}_g(X, Z)$; this is intuitively true because N does not give any information about Y. This proves the upper bound. The lower bound follows from Lemma 7.11 as X(t) is a projection of $(X(t), Z_{\alpha,\beta})$ that omits the last coordinate. This completes the proof of Lemma 7.19.

Using Lemma 7.19, it follows that in this case

$$\operatorname{AUC}_a(X(t), Z_{\alpha,\beta}) \ge \operatorname{AUC}_a(X(t)) > \operatorname{AUC}_b(X(t), Z) \ge \operatorname{AUC}_b(X(t), Z_{\alpha,\beta}).$$

Hence,

$$\begin{aligned} \mathsf{Bias}_{\alpha,\beta}(t+1) &= 1 - \frac{\mathrm{AUC}_b(X(t), Z_{\alpha,\beta})}{\mathrm{AUC}_a(X(t), Z_{\alpha,\beta})} \\ &\geq 1 - \frac{\mathrm{AUC}_b(X(t), Z)}{\mathrm{AUC}_a(X(t))} \end{aligned}$$

(Using Lemma 7.19 to lower bound the denominator and upper bound the numerator)

$$= \operatorname{Bias}(t) - \frac{\operatorname{AUC}_b(X(t), Z) - \operatorname{AUC}_b(X(t))}{\operatorname{AUC}_a(X(t))}.$$
(67)

Further, $\operatorname{Bias}_{\alpha,\beta}(t+1) \ge 0 = \operatorname{Bias}(t) - \operatorname{Bias}(t)$. Combining this inequality with Equation (67) it follows that $\operatorname{Bias}_{\alpha,\beta}(t+1) \ge 0 = \operatorname{Bias}(t) - \Delta$.

This completes the proof of the existence of λ claimed in Theorem 7.17. It remains to prove the weak Pareto-optimality condition. Toward this observe that

$$\operatorname{AUC}_{b}(X(t), Z_{\alpha, \beta}) \stackrel{\text{Lemma 7.19}}{\leq} \operatorname{AUC}_{b}(X(t), Z) \stackrel{(58)}{=} \operatorname{AUC}_{b}(X(t), Z_{\lambda}).$$
(68)

It remains to prove that $AUC_a(X(t), Z_{\alpha,\beta}) \leq AUC_a(X(t), Z_{\lambda})$.

To prove this, we will use the fact that the λ constructed in the proof of Corollary 7.15 satisfies

$$\operatorname{AUC}_{a}(X(t), Z_{\lambda}) \ge \operatorname{AUC}_{b}(X(t), Z_{\lambda}).$$
 (69)

Case A (AUC_a(X(t), Z_{α,β}) \geq AUC_b(X(t), Z_{α,β})): Note that if Bias_{α,β}(t + 1) = Bias_{λ}(t + 1) = 1, then

$$\operatorname{AUC}_{a}(X(t), Z_{\alpha, \beta}) = \operatorname{AUC}_{b}(X(t), Z_{\alpha, \beta}) \stackrel{(68)}{\leq} \operatorname{AUC}_{b}(X(t), Z_{\lambda}) = \operatorname{AUC}_{a}(X(t), Z_{\lambda}).$$

Hence, we are done if $\operatorname{Bias}_{\alpha,\beta}(t+1) = \operatorname{Bias}_{\lambda}(t+1) = 1$. Thus, assume that $\operatorname{Bias}_{\alpha,\beta}(t+1) = \operatorname{Bias}_{\lambda}(t+1) < 1$. Because of the assumption in this case (i.e., Case A), we have that $\operatorname{Bias}_{\alpha,\beta}(t+1) = 1 - \frac{\operatorname{AUC}_{b}(X(t),Z_{\alpha,\beta})}{\operatorname{AUC}_{a}(X(t),Z_{\alpha,\beta})}$. Hence,

$$\begin{aligned} \operatorname{AUC}_{a}(X(t), Z_{\alpha,\beta}) &= \frac{\operatorname{AUC}_{b}(X(t), Z_{\alpha,\beta})}{1 - \operatorname{Bias}_{\alpha,\beta}(t+1)} & (\operatorname{Using that } \operatorname{Bias}_{\alpha,\beta}(t+1) < 1) \\ &\stackrel{\operatorname{Lemma}}{\leq} ^{7.19} \frac{\operatorname{AUC}_{b}(X(t), Z)}{1 - \operatorname{Bias}_{\alpha,\beta}(t+1)} \\ &= \frac{\operatorname{AUC}_{b}(X(t), Z)}{1 - \operatorname{Bias}_{\lambda}(t+1)} & (\operatorname{Using that, by construction, } \operatorname{Bias}_{\lambda}(t+1) = \operatorname{Bias}_{\alpha,\beta}(t+1)) \\ &\stackrel{(58)}{\leq} \frac{\operatorname{AUC}_{b}(X(t), Z_{\lambda})}{1 - \operatorname{Bias}_{\lambda}(t+1)} \\ &= \operatorname{AUC}_{a}(X(t), Z_{\lambda}). & (\operatorname{Using Equation (69) and definition of } \operatorname{Bias}_{\lambda}(t+1)) \end{aligned}$$

Case B (AUC_a(X(t), $Z_{\alpha,\beta}) < AUC_b(X(t), Z_{\alpha,\beta})$): In this case,

$$\operatorname{AUC}_{a}(X(t), Z_{\alpha, \beta}) < \operatorname{AUC}_{b}(X(t), Z_{\alpha, \beta}) \stackrel{(68)}{\leq} \operatorname{AUC}_{b}(X(t), Z_{\lambda}) \stackrel{(69)}{\leq} \operatorname{AUC}_{a}(X(t), Z_{\lambda}).$$

While this noisy fairAUC procedure prevents bias from increasing round to round, introducing noise reduces the AUC of the would-be advantaged group. Therefore the original fairAUC procedure Pareto dominates the noisy fairAUC procedure in terms of AUCs.

8 Additional Empirical Results

8.1 fairAUC with a Bias Penalty Term

Another potential strategy to prevent bias from increasing is to add a penalty term for bias. However, we find that such a strategy is unable to ensure bias does not increase after acquiring a new feature. Figures 7 (Left) and (Right) show the performance of fairAUC with different levels of penalty placed on bias. In Figure 7 (Left), equal weights are placed on AUC and the bias penalty and we can see that the bias still can increase after the acquisition of a feature with fairAUC. In Figure 7 (Right), greater weight is placed on the bias penalty term. Placing a large penalty on bias essentially reduces the procedure to the minBias procedure.

Figure 7: (Left) AUC and Bias Penalty Equally Weighted. (Right) 33% Weight on AUC and 67% Weight on Bias Penalty.



Note: R1 represents Round 1 of feature acquisition and R10 represents Round 10.

Noisy fairAUC prevents bias from increasing each iteration but a penalty-based strategy is not sufficient.

8.2 Empirical Results When the Protected Attribute is Not Used in Classification

fairAUC does not require the use of the protected attribute during classification. In the main body of the paper, the protected attribute was used during classification. Here we show how fairAUC performs when the protected attribute is not used.

8.2.1 Synthetic Data Analysis

As can be seen in Figures 8, 9, and 10, the same pattern of results continues to hold when the protected attribute is not used in classification. Compared to maxAUC, the fairAUC procedure greatly decreases the bias between the two groups and improves the AUC of the disadvantaged group. minBias also greatly reduces bias but at the cost of learning about either group. While fairAUC decreases bias relative to maxAUC, it does trade off AUC in the process as shown in Figure 9.

Figure 8: Group-wise AUCs over Feature Augmentation Rounds without Protected Attribute



Figure 9: Accuracy-Fairness Tradeoff without Protected Attribute



Figure 10: Pareto Frontier without Protected Attribute



8.2.2 COMPAS Data Analysis

We next show the performance of the various procedures when the protected attribute is not used in classification on the COMPAS dataset in Figure 11. The procedures follow the same pattern as when the protected attribute is used in classification (as seen in Figure 4 in the paper).



Figure 11: Predicting Violent Recidivism without Protected Attribute (Age)

8.3 Synthetic Data with Multivariate Gamma Distribution

In the main analysis, we generate data that follow multivariate normal distributions. To test the robustness of fairAUC to other data generating processes, we generate data that follow different multivariate gamma distributions. We choose the gamma distribution because it can be flexibly parametrized using its shape and rate parameters. The objective here is to investigate whether our theoretical guarantees, which hold when the features are generated from a multivariate normal distribution, continue to hold when the data is instead generated from a gamma distribution.

Gamma distributions are defined by the probability density function:

$$f(x;\alpha,\beta) = \frac{x^{\alpha-1}e^{-\beta x}\beta^{\alpha}}{(\alpha-1)!} \text{ for } x,\alpha,\beta > 0.$$

Parameterized by two parameters, α (shape) and β (rate), gamma distributions can take on a wide variety of distribution shapes. The mean of a gamma distribution is α/β and the variance is α/β^2 . We generate data for two groups such that they have the same shape and rate parameters but one group captures a much larger fraction of observations, mirroring the procedure followed in the main paper for multivariate normal data. The following procedure is used to generate the data:

- 1. For each group, generate p informative features that have the same class-conditional variance but different class-conditional means and a random correlation structure (Pourahmadi and Wang 2015)
- 2. For each group, generate q uninformative features that have no difference in class-conditional means or variances and which have a random correlation structure

Below we show the results of the fairAUC, maxAUC, minBias, and random procedures on two datasets generated by the above procedure with different shape and rate combinations. Table 3 lists the various parameters used and why we choose those parameters. We use different distribution shapes to test the robustness of fairAUC. We train the four procedures using separate (logistic regression) classifiers for each group. Figures 12 and 13 compare the group-wise AUCs over feature acquisition rounds.

The different procedures follow the same pattern observed with the multivariate normal distributions. Specifically, we observe that fairAUC reduces the bias to a very small degree, where the two groups a and b have similar AUCs, compared to maxAUC, where group a has a higher AUC, and the overall bias is high. fairAUC selects features that improve the AUC for the disadvantaged group relative to maxAUC. The minBias algorithm obtains low bias, but obtains a really low AUC for each group.

Parameter	Value	Reason for Choice
Fraction of observations in group a	0.95	Large group imbalance as is often seen in real data
Number of observations	20,000	Larger datasets give us more precise AUC results
Dataset 1:		
Negative class α and β	$1,\sqrt{2}$	Hump-shaped with long right tail so distributions
Positive class α and β	2, 2	different from normal but still hump-shaped
Dataset 2:		
Negative class α and β	1, 0.5	Negative class exponential distribution-shaped so
Positive class α and β	$2, \sqrt{0.5}$	deviates more greatly from normal distribution; positive class hump-shaped with long right tail

Table 3: Parameters used for Synthetic Gamma Data



Figure 12: Performance for Dataset 1





8.4 Nonlinear Classifiers

The theoretical guarantees obtained in the paper hold for generalized linear models (GLMs). The experiments with synthetic data and empirical application with real-world data in the main body of the paper use logistic regression as the classifier. Here, we examine the robustness of the results to using nonlinear classifiers, specifically *random forest* and *support vector machine* (SVM) with a nonlinear (rbf) kernel. We use the protected attribute during classification.

Random Forest: Figure 14 shows the performance of the four procedures on multivariate normal data (Guyon 2003) using a random forest classifier. We limit the maximum depth of the forest to three layers to prevent model overfitting. The pattern of results is consistent with the results from the logistic regression model, which are detailed in Figure 3. For example, fairAUC selects features that improve the AUC of the disadvantaged group and in doing so reduces bias relative to maxAUC. minBias fails to acquire informative features.



Figure 14: Performance using Random Forest as Classifier

Next, we compare which features are acquired depending on whether logistic regression or random forest is used in Table 4. Note that logistic regression and FLD select the same features in the same order on this data set over the first ten rounds. For nine out of the first ten rounds, logistic regression and random forest acquire the same features. We highlight in bold which features differ between the two classifiers. Random forest generates slightly higher AUCs.

Table 4: Using Logistic Regression vs. Random Forest for Classification

			Logistic 1	Regression					Randor	n Forest		
Round	Feature	AUC_a	AUC_b	AUCAll	Bias	Disadv	Feature	AUC_a	AUC_b	AUCAll	Bias	Disadv
						Group						Group
0		0.6058	0.5083	0.5846	0.1610	b		0.6217	0.6253	0.6127	0.0057	a
1	47	0.6063	0.6946	0.6368	0.1271	a	18	0.7372	0.6278	0.7056	0.1484	b
2	18	0.7297	0.6976	0.7205	0.0439	b	47	0.7393	0.7344	0.7357	0.0066	b
3	13	0.7299	0.7266	0.7291	0.0045	b	13	0.7441	0.7612	0.7482	0.0225	a
4	48	0.7300	0.7486	0.7359	0.0249	a	26	0.7891	0.7573	0.7797	0.0403	b
5	26	0.7878	0.7494	0.7771	0.0488	b	48	0.7890	0.7827	0.7868	0.0079	b
6	30	0.7879	0.7736	0.7840	0.0181	b	30	0.7893	0.7956	0.7909	0.0079	a
7	15	0.7879	0.7861	0.7877	0.0023	b	20	0.8189	0.7921	0.8095	0.0327	b
8	49	0.7880	0.8048	0.7934	0.0209	a	15	0.8139	0.8082	0.8119	0.0070	b
9	20	0.8209	0.8049	0.8163	0.0194	b	12	0.8199	0.8140	0.8175	0.0073	b
10	12	0.8209	0.8130	0.8186	0.0095	b	38	0.8212	0.8227	0.8211	0.0018	a

SVM with Nonlinear Kernel: Figure 15 shows the performance of the four procedures on multivariate normal data (Guyon 2003) using SVM with a nonlinear kernel function. Again, the pattern of results is consistent with the logistic regression results. Table 5 compares which features are acquired depending on whether logistic regression or nonlinear SVM is used. Nonlinear SVM results in much higher AUC values than logistic regression but both end up acquiring roughly the same set of features in the first ten rounds (eight out of ten features overlap).



Figure 15: Performance using Nonlinear SVM as Classifier

Table 5: Using Logistic Regression vs. Nonlinear SVM for Classification

			Logistic 1	Regression					Nonline	ear SVM		
Round	Feature	AUC_a	AUC_b	AUC_{All}	Bias	Disadv	Feature	AUC_a	AUC_b	AUC _{All}	Bias	Disadv
						Group						Group
0		0.6058	0.5083	0.5846	0.1610	b		0.5140	0.4983	0.5122	0.0305	b
1	47	0.6063	0.6946	0.6368	0.1271	a	47	0.5415	0.5965	0.5610	0.0922	a
2	18	0.7297	0.6976	0.7205	0.0439	b	18	0.6819	0.6479	0.6657	0.0498	b
3	13	0.7299	0.7266	0.7291	0.0045	b	13	0.7225	0.7257	0.7221	0.0045	a
4	48	0.7300	0.7486	0.7359	0.0249	a	26	0.7912	0.7372	0.7713	0.0683	b
5	26	0.7878	0.7494	0.7771	0.0488	b	49	0.8168	0.8045	0.8137	0.0151	b
6	30	0.7879	0.7736	0.7840	0.0181	b	38	0.8215	0.8456	0.8290	0.0285	a
7	15	0.7879	0.7861	0.7877	0.0023	b	17	0.8663	0.8690	0.8671	0.0031	a
8	49	0.7880	0.8048	0.7934	0.0209	a	20	0.9026	0.8727	0.8938	0.0331	b
9	20	0.8209	0.8049	0.8163	0.0194	b	12	0.9239	0.8946	0.9157	0.0317	b
10	12	0.8209	0.8130	0.8186	0.0095	b	15	0.9251	0.9122	0.9214	0.0139	b

The results suggest that the FLD heuristic for feature acquisition is robust to using different classifiers.

8.5 Acquiring Multiple Features at a Time

The fairAUC algorithm can certainly be altered to acquire more than one feature at a time. The benefit to acquiring more than one feature at a time would be that the algorithm is then able to incorporate the covariance between the auxiliary features. However, acquiring multiple features also increases the complexity of the problem and decreases the flexibility in determining which group is the disadvantaged group. If there are n auxiliary features and we plan to acquire k features at a time, there are $\binom{n}{k}$ possible combinations of features to acquire, which grows faster than 2^k for $k < \frac{n}{2}$.

We show the AUC and bias when fairAUC is used to collect two features each feature acquisition round. We use Equation 7 to calculate the AUC associated with the acquisition of each possible pair of auxiliary features. Table 6 compares the AUCs and bias that result from acquiring one feature at a time versus acquiring two features at a time on the synthetic dataset generated using Guyon (2003). Group a begins as the disadvantaged group. First, we observe that nine out of the ten features first acquired are the same but in slightly different orders. Second, acquiring two features at a time results in slightly higher overall AUCs across both groups. Third, acquiring one feature at a time results in lower bias on average (0.0395 for one feature vs. 0.0704 for two features). This lower bias is likely because choosing one feature at a time is less likely to overshoot on AUC since it allows flexibility in determining which is the disadvantaged group in each round. Increasing the number of features to acquire k would further decrease flexibility and allow for greater overshooting of AUC.

			One F	Peature			Two Features					
Round	Feature	AUC_a	AUC_b	AUC _{All}	Bias	Disadv	Features	AUC_a	AUC_b	AUC _{All}	Bias	Disadv
						Group						Group
0		0.5057	0.5810	0.5332	0.1296	a		0.5057	0.5810	0.5332	0.1296	a
1	17	0.6825	0.6298	0.6679	0.0772	b						
2	47	0.6832	0.7247	0.6965	0.0572	a	17,20	0.7547	0.6309	0.7237	0.1641	b
3	20	0.7549	0.7251	0.7463	0.0395	b						
4	12	0.7550	0.7599	0.7566	0.0065	a	13,47	0.7551	0.7599	0.7566	0.0064	a
5	26	0.7932	0.7601	0.7839	0.0417	b						
6	13	0.7946	0.7831	0.7913	0.0144	b	18,26	0.8302	0.7614	0.8114	0.0829	b
7	15	0.7946	0.7992	0.7962	0.0057	a						
8	18	0.8303	0.8000	0.8217	0.0366	b	12,15	0.8303	0.8000	0.8217	0.0366	b
9	49	0.8336	0.8181	0.8292	0.0186	b						
10	38	0.8337	0.8278	0.8322	0.0071	b	7,49	0.8336	0.8312	0.8333	0.0029	b

Table 6: Selecting One Feature at a Time vs. Two at a Time using fairAUC

One strategy to decrease the complexity of the problem but still allow for more than one feature to be acquired at a time is to first acquire the "best" feature and then acquire the feature least correlated with the "best" feature. The "best" feature is the feature which increases the AUC of the disadvantaged group the most. We use the unconditional correlation to determine which second auxiliary feature should be acquired given the first. Once the first feature is determined, there are only n-1 combinations of two features.

Table 7 shows the AUCs and bias of selecting two features simultaneously versus sequentially according to the strategy proposed above (i.e., best feature v and least correlated with v feature). Selecting two features simultaneously results in higher AUCs.

		Si	multaneou	18		Sequential				
Round	Features	AUC_a	AUC_b	AUC _{All}	Bias	Features	AUC_a	AUC_b	AUCAll	Bias
0		0.5057	0.5810	0.5332	0.1296		0.5057	0.5810	0.5332	0.1296
1	17,20	0.7547	0.6309	0.7237	0.1641	17,37	0.6827	0.6317	0.6685	0.0747
2	13,47	0.7551	0.7599	0.7566	0.0064	47,10	0.7034	0.7256	0.7104	0.0305
3	18,26	0.8302	0.7614	0.8114	0.0829	20,24	0.7717	0.7267	0.7591	0.0584
4	12,15	0.8303	0.8000	0.8217	0.0366	12,32	0.7721	0.7630	0.7697	0.0119
5	7,49	0.8336	0.8312	0.8333	0.0029	13,8	0.7723	0.7865	0.7769	0.0180

Table 7: Selecting Two Features Simultaneously vs. Sequentially

8.6 Data Vendor Features

We acquire auxiliary features to augment the COMPAS dataset from Aspire North²³, a data vendor. Table 8 lists some of the features that the firm offers. The firm sells a core bundle of features for 80/1,000 individuals. Certain variables, like ethnicity, net worth, spending, cost more and purchasing all available variables exceeds 15,000/1,000 individuals.

To obtain data from the vendor, we must provide identifying information on the customers, which can include the following types of data to establish identity: name, date of birth, email address, and home address.

²³https://www.aspire-north.com/

Feature Examples	Cost/1,000 individuals	# of Features
		Available
Age, gender, education, marital status	Core bundle	26
House type, renter, homeowner	Core bundle	14
Vehicle ownership, computer owner-	Core bundle	23
ship		
Interest in crafts, gourmet cooking	Core bundle	55
Estimate of deposit, balances	\$40 per feature	5
Prediction of annual, spend on dining	\$50 per feature	10
Prediction of language preference	\$16 per feature	6
Prediction of preference for rewards	\$400 per feature	37
credit cards		
Prediction of household spend on edu-	\$50 per feature	15
cation		
	Age, gender, education, marital status Age, gender, education, marital status House type, renter, homeowner /ehicle ownership, computer owner- hip nterest in crafts, gourmet cooking Estimate of deposit, balances Prediction of annual, spend on dining Prediction of language preference Prediction of preference for rewards credit cards Prediction of household spend on edu- cation	reature Examples Cost/1,000 individuals Age, gender, education, marital status Core bundle House type, renter, homeowner Core bundle /ehicle ownership, computer owner- Core bundle hip Core bundle Stimate of deposit, balances \$40 per feature Prediction of annual, spend on dining \$50 per feature Prediction of preference for rewards \$400 per feature Prediction of household spend on edu- \$50 per feature \$50 per feature \$50 per feature

Table 8: Aspire North Feature Examples

8.7 Using Only Class-Conditional Means and Variances

The fairAUC procedure requires data sharing between the firm and data vendor because of the classconditional correlations between the score (which the firm manager has access to) and the auxiliary features (which the data vendor has access to). One strategy to simplify the procedure even further is to assume independence between the auxiliary features and the firm's data. While this assumption is likely to be incorrect, we evaluate how well fairAUC performs in such a case. If we are willing to make such an assumption, then the data vendor can just provide means and variances corresponding to specific individuals chosen by the firm, without transferring the score or any other data about the individuals except for class.

Table 9 shows the differences in AUC between the procedure that includes the class-conditional correlations and that which assumes the correlations to be zero. Specifically, we subtract the AUC with zero correlations from the AUC with correlations and do the same for the bias values. We highlight in bold which features are acquired when the correlations are assumed to be zero but are not acquired when the correlations are accounted for. Only two features fall into this category. However, incorporating the class-conditional correlations generally results in higher overall AUC and lower bias, as we might expect, which illustrates the tradeoff. Therefore, broadly, it seems like ignoring the class-conditional correlations generally results in the acquisition of the same features but in perhaps a less efficient order.

	Feature with	Feature without	Difference	Difference	Difference	Difference
Round	Correlations	Correlations	in AUC_a	in AUC_b	in AUC _{All}	in Bias
1	17	17	0	0	0	0
2	47	47	0	0	0	0
3	20	18	0.0081	-0.0011	0.0055	0.0120
4	12	13	0.0079	-0.0005	0.0054	-0.0111
5	26	26	-0.0017	-0.0008	-0.0012	-0.0010
6	13	38	-0.0005	0.0114	0.0031	-0.0149
7	15	29	-0.0006	0.0185	0.0053	-0.0125
8	18	49	0.0313	-0.0005	0.0220	0.0347
9	49	20	-0.0001	0.0174	0.0047	-0.0209
10	38	28	0.0006	0.0196	0.0058	-0.0228

Table 9: Effect of Ignoring Conditional Correlation between Acquired Data and Auxiliary Features

Note: Difference = (with correlation) - (without correlation)

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A Measures of Fairness Used in the Literature

Table 10 provides advantages and disadvantages to several fairness metrics that have been suggested in the literature. Let $Y \in \{0, 1\}$ represent the true outcome, $\hat{Y} \in \{0, 1\}$ the predicted outcome, A the protected attribute, X the non-protected attributes, and C the classifier.

Table 10: Measures of Fairness in the Literature

Measure	Definition, Advantages, & Disadvantages
Unawareness	C = C(X) Advantage: Addresses disparate treatment and complies with existing laws (e.g., Civil Rights Act of 1964) by not using protected attribute as an explicit variable. Disadvantage: If X and A are correlated then the protected attribute is still incorporated into the classifier.
Statistical Parity	$\Pr[\hat{Y} = 1 A = i] = \Pr[\hat{Y} = 1 A = j]$ for all groups <i>i</i> and <i>j</i> Advantage: Addresses disparate impact and is the foundation for some laws (e.g., four-fifths rule). Disadvantages: Can be achieved simply by selecting $x\%$ from all groups regardless of justi- fication, potentially resulting in reverse-discrimination. If <i>Y</i> and <i>A</i> are correlated, the ideal predictor $\hat{Y} = Y$ cannot be obtained.
Predictive Rate Parity	$\Pr[Y = 1 A = i, \hat{Y} = 1] = \Pr[Y = 1 A = j, \hat{Y} = 1]$ for all groups i and j and $\Pr[Y = 0 A = i, \hat{Y} = 0] = \Pr[Y = 0 A = j, \hat{Y} = 0]$ for all groups i and j Advantage: Optimality-compatible (i.e., allows $\hat{Y} = Y$), aligning fairness with accuracy, and avoids reverse-discrimination. Disadvantage: May not close the gap between groups over time if Y and A are correlated.
Equalized Odds	$\Pr[\hat{Y} = 1 A = i, Y = 1] = \Pr[\hat{Y} = 1 A = j, Y = 1]$ for all groups i and j and $\Pr[\hat{Y} = 1 A = i, Y = 0] = \Pr[\hat{Y} = 1 A = j, Y = 0]$ for all groups i and j Advantage: Optimality-compatible (i.e., allows $\hat{Y} = Y$) and avoids reverse-discrimination. Disadvantage: May not close the gap between groups over time if Y and A are correlated.

B maxAUC Algorithm

Procedure 2: maxAUC (*t*-th iteration)

Input: data owned $(\mathbf{X}_i, A_i, Y_i)_{i=1}^N$, scoring algorithm r, bias threshold ε , set of acquired features Q(t), data available for acquisition $(\mathbf{Z}_i)_{i=1}^N$; **Output:** Q(t+1);if A cannot be used then $\boldsymbol{S} \coloneqq r(\hat{\boldsymbol{X}}, \boldsymbol{Y});$ else $| \quad \boldsymbol{S} := r(\hat{\boldsymbol{X}}, \boldsymbol{A}, \boldsymbol{Y});$ for group $g \in \{a, b\}$ do compute $AUC_g(S)$ (Definition 3.1); $\text{Bias} \coloneqq 1 - \frac{\min_g(\text{AUC}_g(S))}{\max_g(\text{AUC}_g(S))} \text{ (Definition 3.2) };$ $\mathbf{if}\ \mathrm{Bias} > \varepsilon\ \mathbf{then}$ for feature $\mathbf{Z}^j \in \hat{\mathbf{Z}}, j \notin Q(t)$ do if A cannot be used then for feature Z^{j} and score S, obtain class-conditional means, μ_{0}, μ_{1} , and covariance matrices, Σ_{0}, Σ_{1} (Overall Summary Statistics Subroutine); $h(\boldsymbol{S}, \boldsymbol{Z}^j) \coloneqq \Phi\left(\sqrt{(\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)^\top (\boldsymbol{\Sigma}_0 + \boldsymbol{\Sigma}_1)^{-1} (\boldsymbol{\mu}_1 - \boldsymbol{\mu}_0)}\right);$ else for $g \in \{a, b\}$ do for feature Z^{j} , group g, and score S, obtain class-conditional means, μ_{0}, μ_{1} , and covariance matrices, Σ_0, Σ_1 (Summary Statistics by Group Subroutine); ϕ_g represents the proportion of individuals from group g; $h(\boldsymbol{S}, \boldsymbol{Z}^j) := \phi_a \Phi(\sqrt{\omega_a}) + \phi_b \Phi(\sqrt{\omega_b});$ $j^{\star} \coloneqq \arg \max_{j} h(\boldsymbol{S}, \boldsymbol{Z}^{j});$ acquire feature Z^{j^*} ; return $Q(t+1) := Q(t) \cup \{j^\star\};$ else no intervention;

Subroutine: Overall Summary Statistics

Input: feature available for acquisition Z, existing score S; Output: class-conditional mean vectors μ_0, μ_1 , class-conditional covariance matrices Σ_0, Σ_1 ; for class $y \in \{0, 1\}$ do $n \coloneqq n_{Y=y}$; $\mu_y \coloneqq \left[\frac{\bar{S}_y}{\bar{Z}_y}\right] = \left[\frac{\frac{1}{n}\sum_{i:Y_i=y}S_i}{\frac{1}{n}\sum_{i:Y_i=y}Z_i}\right]$; $\Sigma_y \coloneqq \left[\frac{\sigma_{S,y}^2 \quad \rho_y\sigma_{S,y}\sigma_{Z,y}}{\rho_y\sigma_{S,y}\sigma_{Z,y} \quad \sigma_{Z,y}^2}\right]$ where $\sigma_{S,y}^2 = \frac{1}{n-1}\sum_{i:Y_i=y}(S_i - \bar{S}_y)^2, \sigma_{Z,y}^2 = \frac{1}{n-1}\sum_{i:Y_i=y}(Z_i - \bar{Z}_y)^2$, and $\rho_y = \frac{1}{(n-1)\sigma_{s,y}\sigma_{Z,y}}\sum_{i:Y_i=y}(S_i - \bar{S}_y)(Z_i - \bar{Z}_y)$; return $\mu_0, \mu_1, \Sigma_0, \Sigma_1$